

PARABOLIC SHEAVES WITH REAL WEIGHTS AS SHEAVES ON THE KATO-NAKAYAMA SPACE

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ABSTRACT. We define quasi-coherent parabolic sheaves with real weights on a fine saturated log analytic space, and explain how to interpret them as quasi-coherent sheaves of modules on its Kato-Nakayama space. This recovers the description as sheaves on root stacks of [5] and [22] for rational weights, but also includes the case of arbitrary real weights.

CONTENTS

1.	Introduction	1
2.	Preliminaries	4
3.	Parabolic sheaves with real weights	6
4.	Sheaves on the Kato-Nakayama space	14
5.	The correspondence	25
	References	32

1. INTRODUCTION

The aim of this paper is to present a correspondence between parabolic sheaves with real weights on a fine saturated log analytic space, and certain sheaves of modules on its Kato-Nakayama space. This was inspired by the corresponding equivalence for rational weights and root stacks [5, 22], and by the analogy between the infinite root stack and the “profinite completion” of the Kato-Nakayama space [6, 23].

Parabolic bundles were first defined by Metha and Seshadri on curves in the ’80s [16], and then studied in increasingly more general situations by several authors [14, 17, 10, 4], until Borne and Vistoli [5] linked them with logarithmic structures, and gave a general definition (for rational weights with bounded denominator) on a coherent log scheme. They also constructed an equivalence of abelian categories between parabolic sheaves with weights in a fixed Kummer extension, and quasi-coherent sheaves on the corresponding stack of roots. Versions of this correspondence were earlier investigated by Biswas [1] and Borne [4], and it was further generalized to arbitrary rational weights by the author and Vistoli in [22].

Assume for simplicity in this introduction that X is a scheme of finite type over \mathbb{C} , whose log structure is determined by a single effective Cartier divisor $D \subseteq X$, or, equivalently, by the line bundle with section $(L, s) := (\mathcal{O}_X(D), 1_D)$ (this data also gives a log analytic space, by analytifying X and D). In the algebraic setting, parabolic sheaves with weights in the group $\frac{1}{n}\mathbb{Z}$ are sequences of quasi-coherent sheaves $\{E_a\}$ on X for $a \in \frac{1}{n}\mathbb{Z}$, with: a system of compatible maps $E_a \rightarrow E_b$ every time $b \geq a$, isomorphisms $E_{a+1} \cong E_a \otimes_{\mathcal{O}_X} L$ for all a , and such that $E_a \rightarrow E_{a+1} \cong E_a \otimes_{\mathcal{O}_X} L$ coincides with multiplication by $s \in \Gamma(L)$. Clearly, such an object is completely determined by its restriction to the segment $[0, 1]$, i.e. the diagram

$$E_0 \longrightarrow E_{\frac{1}{n}} \longrightarrow \cdots \longrightarrow E_{\frac{n-1}{n}} \longrightarrow E_1 \cong E_0 \otimes_{\mathcal{O}_X} L.$$

For a general fine saturated log scheme, a parabolic sheaf is also a system of sheaves with maps, indexed by a locally constant sheaf of (possibly higher-rank) lattices.

The root stack $\sqrt[n]{X}$ parametrizes roots of the pair (L, s) , i.e. a morphism $T \rightarrow \sqrt[n]{X}$ corresponds to a map $f: T \rightarrow X$ and a pair (N, t) on T consisting of a line bundle with a section, with an isomorphism $(N, t)^{\otimes n} \cong f^*(L, s)$. There is a coarse moduli space morphism $\pi: \sqrt[n]{X} \rightarrow X$, which is an isomorphism outside of D . Points in the preimage of D have a non-trivial stabilizer, the group of n -th roots of unity μ_n . The main result of [5], in this particular case, says that there is an equivalence of abelian categories between parabolic sheaves with weights in $\frac{1}{n}\mathbb{Z}$ and quasi-coherent sheaves on $\sqrt[n]{X}$.

The functor $\Phi_n: \text{Qcoh}(\sqrt[n]{X}) \rightarrow \text{Par}(X, \frac{1}{n}\mathbb{Z})$ is easily described as follows: for a given $F \in \text{Qcoh}(\sqrt[n]{X})$, one sets $\Phi_n(F)_{\frac{1}{n}k} := \pi_*(F \otimes_{\mathcal{O}_{\sqrt[n]{X}}} \mathcal{N}^{\otimes k})$, where \mathcal{N} is the universal “root line bundle” on $\sqrt[n]{X}$. Moreover, for $\frac{1}{n}k \leq \frac{1}{n}k'$, there is a natural map $\mathcal{N}^{\otimes k} \rightarrow \mathcal{N}^{\otimes k'}$ given by the appropriate power of the global section t of \mathcal{N} , that induces a morphism $\Phi_n(F)_{\frac{1}{n}k} \rightarrow \Phi_n(F)_{\frac{1}{n}k'}$. The projection formula for π assures that the other properties in the definition of a parabolic sheaf are satisfied. Heuristically, the presence of the non-trivial stabilizers μ_n along the divisor (and its action on fibers of sheaves) allows to encode the different pieces of the parabolic sheaves in a single sheaf on the root stack.

If we allow the index of the root to vary, these equivalences are compatible with the natural projections $\sqrt[n]{X} \rightarrow \sqrt[m]{X}$ for $n \mid m$, and in fact there is an analogous statement at the limit, on the infinite root stack $\sqrt[\infty]{X}$ [22, Theorem 7.3]. This “stacky” point of view allows to treat parabolic sheaves as “plain” quasi-coherent sheaves on a slightly more complicated object, and has been useful in several instances (see for example [9], [2] and [21]).

In the original definition of Metha and Seshadri, as well as in later instances, parabolic sheaves are allowed to have arbitrary real weights. In the situation of a scheme X with a divisor D as above, a parabolic sheaf with real weights is going to be a system of indexed sheaves as in the rational case, but the index group is the set of real numbers \mathbb{R} . Finitely presented sheaves (appropriately defined) will be still determined by finitely many sheaves E_r for $r \in [0, 1]$ and the maps between them, but for general quasi-coherent sheaves, this is not the case.

As can certainly be expected, irrational weights are hard to handle in a purely algebraic manner. In this paper, we extend to real weights the correspondence with sheaves on root stacks, but using the Kato-Nakayama space instead. This forces us to work over the complex numbers.

Recall that the Kato-Nakayama space X_{\log} is a topological space with a continuous proper projection $X_{\log} \rightarrow X$, where X is now a (fine saturated) log analytic space. Morally, this construction replaces the log structure of X with non-trivial topology in X_{\log} . For example, in the situation above, where the log scheme is determined by a single smooth divisor $D \subseteq X$ in a smooth analytic space X , the space X_{\log} is the “real oriented blowup” of D in X .

The use of the Kato-Nakayama space X_{\log} is heuristically justified by the fact that the infinite root stack is a sort of “profinite algebraic incarnation” of the former: there is a morphism $X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$ to the topological realization of $\sqrt[\infty]{X}$, which is a “profinite equivalence” [6, Theorem 6.4]. The fiber of $X_{\log} \rightarrow X$ over a point x can be identified with $(S^1)^r$, and the fiber of $\sqrt[\infty]{X}_{\text{top}} \rightarrow X$ with $B\widehat{\mathbb{Z}}^r$, where r is the “rank of the log structure” at x . Thinking of S^1 as $B\mathbb{Z}$, the morphism between the fibers $B\mathbb{Z}^r \rightarrow B\widehat{\mathbb{Z}}^r \cong \widehat{B\mathbb{Z}}^r$ is the map to the profinite completion. Morally, while the profinite monodromy (i.e. stabilizer group) in the fibers of $\sqrt[\infty]{X} \rightarrow X$ can only allow for rational weights in

the parabolic sheaves, the fibers of $X_{\log} \rightarrow X$ have “monodromy” (i.e. fundamental group) with elements of infinite order, and the S^1 s in the fibers can also encode real weights.

Assume that we are still in the simple situation of a log structure given by a divisor $D \subseteq X$ outlined above, and fix a submonoid Λ of \mathbb{R}_+ , the non-negative real numbers, containing \mathbb{N} . In order to make the heuristic of the previous paragraph precise, we adapt a procedure of Ogus [18] to construct on X_{\log} a sheaf of rings \mathcal{O}_Λ , that extends the pullback of \mathcal{O}_X by adding sections of the form f^λ , where f is a local equation of D and $\lambda \in \Lambda$ (if $\Lambda = \frac{1}{n}\mathbb{N}$, we are extracting n -th roots, in analogy with root stacks). The intuition for why this can be done, is that passing to X_{\log} somewhat corresponds to extracting a logarithm of these local sections f , and if we have a logarithm we can also define $f^\alpha = \exp(\alpha \log(f))$ for any $\alpha \in \mathbb{R}_+$.

After tensoring \mathcal{O}_Λ over the pullback of \mathcal{O}_X with the “structure sheaf” \mathcal{O}_X^{\log} of X_{\log} (see [8, Section 1]), we obtain a sheaf of rings $\mathcal{O}_\Lambda^{\log}$ on X_{\log} , that allows us to encode parabolic sheaves with weights in Λ as quasi-coherent sheaves. The following is our main result.

Main Theorem (Theorem 5.1). *Let X be a fine saturated log analytic space with log structure $\alpha: M \rightarrow \mathcal{O}_X$, and Λ a quasi-coherent sheaf of monoids, with $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}} = \overline{M} \otimes \mathbb{R}_+$ (here, as usual, \overline{M} denotes the sheaf of monoids M/\mathcal{O}_X^\times).*

Then we have an exact equivalence of categories

$$\mathrm{Qcoh}(\mathcal{O}_\Lambda^{\log}) \cong \mathrm{Par}(X, \Lambda)$$

between quasi-coherent sheaves of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} and quasi-coherent parabolic sheaves on X with weights in Λ .

We remark that quasi-coherence in this setting is a less transparent condition than in the algebraic case (see Remark 3.19 and the discussion in (4.5)). The equivalence restricts to finitely presented sheaves on both sides, that are perhaps more natural objects. Moreover, this equivalence is compatible with the ones for root stacks of [5] and [22] via the natural maps $X_{\log} \rightarrow \sqrt[n]{X}_{\mathrm{top}}$, as we verify in (5.1).

We plan to make use of this equivalence in future work, in at least a couple of directions. First, there are probably interesting interactions between these parabolic structures and integrable logarithmic connections, through Ogus’ version of the Riemann-Hilbert correspondence [18]. The sheaves of rings on X_{\log} that he uses are closely related to the ones we use, and in fact his work on this subject was a fundamental inspiration. Second, the point of view advocated in this paper might be useful to study moduli spaces of parabolic sheaves with arbitrary real weights, and in particular for questions related to the variations of the weights. In moduli problems where there is a stability parameter, very often one has a wall-and-chamber decomposition of the space of possible parameters, and the moduli spaces undergo interesting transformations as the parameter crosses a wall. In the setting of parabolic sheaves, some versions of these questions have been investigated in [3, 24].

Outline. Let us describe the contents of each section of the paper. We begin by briefly recalling some basics about log schemes and log analytic spaces, and the construction of root stacks and Kato-Nakayama spaces in Section 2. In Section 3 we extend the definition of parabolic sheaves on a log scheme of [5] to the case of arbitrary real weights. We also include a brief reminder about the proof of the correspondence with quasi-coherent sheaves on root stacks, that we will adapt to the different context when proving Theorem 5.1. We then proceed in Section 4 to describe how to equip the Kato-Nakayama space X_{\log} of a fine saturated log analytic space X with several sheaves of rings (depending on the monoid encoding the weights), and we discuss quasi-coherent

and finitely presented sheaves on X_{\log} . Finally, Section 5 contains the proof of our main result. We also describe how the correspondence with sheaves on the Kato-Nakayama space is related to the one on root stacks, via the natural map between the two objects.

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Notations and conventions. All monoids will be commutative. The terminology “toric” for a monoid will mean fine, saturated and sharp (and hence torsion-free). If P is a monoid, we will denote by P^{gp} the associated group. We denote by \mathbb{R}_+ the commutative monoid of non-negative real numbers, where the operation is addition, and by $\mathbb{R}_{\geq 0}$ the monoid with the same underlying set, but where the operation is multiplication. If P is a monoid and X is a topological monoid, we will denote by $X(P)$ the topological monoid given by $\text{Hom}(P, X)$.

We will typically use the same symbol for a locally finite type scheme over \mathbb{C} , its associated complex analytic space and the underlying topological space of the latter (i.e. the set of closed points of the scheme), occasionally adding a subscript “an” for analytifications. If P is a finitely generated monoid, we will denote by $\mathbb{C}(P)$ the complex analytic space $(\text{Spec } \mathbb{C}[P])_{\text{an}}$.

Sheaves and stacks on a complex analytic space X will always be sheaves and stacks on the classical analytic site. A quasi-coherent sheaf on a complex analytic space X will be a sheaf of \mathcal{O}_X -modules that can locally be written as a filtered colimit of coherent sheaves, as in [7, Section 2.1]. If (T, \mathcal{O}_T) is a ringed space, we will denote by $\text{Mod}(\mathcal{O}_T)$ the category of sheaves of \mathcal{O}_T -modules on T , and by $\text{Mod}_{\mathcal{O}_T}$ the stack over (the classical site of) T , of sheaves of \mathcal{O}_T -modules. A sheaf of \mathcal{O}_T -modules will be called finitely presented if locally on T it is the cokernel of a morphism of free \mathcal{O}_T -modules of finite rank.

If T is a topological space and G is a topological monoid, or group, etc. we denote by G_T the sheaf of continuous functions towards G on opens of T (with the induced structure of a sheaf of monoids, or groups, etc.). If S is a set, the locally constant sheaf with fiber S on the space T will be denoted by \underline{S}_T . This will also have the induced structure, if S is a monoid, or group, etc.

2. PRELIMINARIES

In this section we briefly recall the basics of log schemes and log analytic spaces, root stacks and the Kato-Nakayama space. For more details, we refer the reader to [6, Appendix] and references therein.

2.1. Log schemes and analytic spaces. A log scheme is a scheme X with a sheaf of monoids M on the small étale site, and a homomorphism $\alpha: M \rightarrow \mathcal{O}_X$, that induces an isomorphism $\alpha|_{\alpha^{-1}\mathcal{O}_X^\times}: \alpha^{-1}\mathcal{O}_X^\times \cong \mathcal{O}_X^\times$. Assuming that M is a sheaf of integral monoids, this additional data is equivalent to a “Deligne–Faltings structure” (abbreviated by DF from now on), i.e. a symmetric monoidal functor $L: A \rightarrow \text{Div}_X$ with trivial kernel (meaning that if $L(a)$ is an invertible object, then $a = 0$), where A is a sheaf of sharp monoids on X and Div_X is the stack of line bundles with a section on the small étale site of X . Given a log structure $\alpha: M \rightarrow \mathcal{O}_X$, the functor L is obtained by modding out in the stacky sense by the action of \mathcal{O}_X^\times (so in particular the sheaf A is $\overline{M} = M/\mathcal{O}_X^\times$).

A morphism of log schemes $f: X \rightarrow Y$ is a morphism of schemes, together with a homomorphism of monoids $f^*M_Y \rightarrow M_X$ that is compatible with the maps to the structure sheaves. A morphism of log schemes is strict if this last homomorphism is an isomorphism (i.e. the log structure of X

is obtained by that of Y by pullback). There is an analogous description of morphisms using DF structures.

If P is a toric monoid, then the scheme $\mathrm{Spec} \mathbb{Z}[P]$ (or $k[P]$ if we are working over a field k) has a canonical log structure, determined by the homomorphism of monoids $P \rightarrow \mathbb{Z}[P]$, after sheafifying and fixing the behaviour of the units. More generally, a Kato chart for the log scheme X is a homomorphism of monoids $P \rightarrow \mathcal{O}_X$ that induces $\alpha: M \rightarrow \mathcal{O}_X$ after sheafifying and fixing the behaviour of the units. Equivalently, it is a strict morphism $X \rightarrow \mathrm{Spec} \mathbb{Z}[P]$.

We will work with fine saturated log schemes, those for which locally for the étale topology we can find charts as above with P integral, finitely generated and saturated (one can moreover take it to be sharp). All of this also applies word for word to complex analytic spaces, where instead of the étale topology we use the classical (analytic) topology. A log structure on a locally finite type scheme X over \mathbb{C} induces a log structure on the analytification X_{an} (see for example [23, Section 2.5]).

2.2. Root stacks. Let X be a fine saturated log scheme or log analytic space. Assume that $\overline{M} \rightarrow B$ is a system of denominators, in the language of [5], i.e. it is an injective map of Kummer type (every section of B locally has a multiple in \overline{M}), and B has local charts by finitely generated monoids (a homomorphism of monoids $Q \rightarrow B(X)$ with Q finitely generated, such that the induced morphism of sheaves $f: Q_X \rightarrow B$ is a cokernel of sheaves of monoids, i.e. $B \cong Q_X / \ker f$). A typical example is the inclusion $\overline{M} \rightarrow \frac{1}{n}\overline{M}$.

The root stack with respect to B , denoted $\sqrt[n]{B}$, is the stack over X parametrizing liftings of $L: \overline{M} \rightarrow \mathrm{Div}_X$ to a symmetric monoidal functor $\frac{1}{n}\overline{M} \rightarrow \mathrm{Div}_X$. It is a tame algebraic stack (Deligne–Mumford in characteristic 0), with a proper quasi-finite coarse moduli space morphism $\sqrt[n]{B} \rightarrow X$. Roughly, $\sqrt[n]{B}$ is the stack obtained by extracting roots out of sections of \overline{M} , with respect to indices dictated by the sections of the sheaf of monoids B (for instance if $B = \frac{1}{n}\overline{M}$, we are extracting n -th roots of all sections of \overline{M}).

Locally where $\overline{M} \rightarrow B$ has a chart $P \rightarrow Q$ (meaning that this is a Kummer homomorphism, $P \rightarrow \overline{M}(X)$ and $Q \rightarrow B(X)$ are charts, and the obvious square commutes), and the chart for \overline{M} is a Kato chart (i.e. it lifts to $P \rightarrow M(X)$), the root stack $\sqrt[n]{B}$ is isomorphic to the quotient stack

$$[\mathrm{Spec}_X(\mathcal{O}_X[P^{\mathrm{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) / \widehat{Q}],$$

where the group $\widehat{Q} = \mathrm{Hom}(Q^{\mathrm{gp}}, \mathbb{G}_m)$ acts on

$$\mathrm{Spec}_X(\mathcal{O}_X[P^{\mathrm{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) \cong \mathrm{Spec}_X(\mathcal{O}_X[P^{\mathrm{gp}}]) \times_{\mathrm{Spec} \mathbb{Z}[P]} \mathrm{Spec} \mathbb{Z}[Q]$$

via the natural grading of the second factor [5, Remark 4.14]. In particular, quasi-coherent sheaves on $\sqrt[n]{B}$ can be identified with quasi-coherent sheaves of $\mathcal{O}_X[P^{\mathrm{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ -modules on X that have a Q^{gp} -grading, compatible with the module structure.

This quotient presentation is more convenient to describe the correspondence with parabolic sheaves, but there is a perhaps simpler one, where the group that we quotient by is finite. Precisely, in presence of a Kato chart as above, there is an isomorphism

$$(1) \quad \sqrt[n]{B} \cong [(X \times_{\mathrm{Spec} \mathbb{Z}[P]} \mathrm{Spec} \mathbb{Z}[Q]) / \mu_{Q/P}],$$

where $\mu_{Q/P}$ is the Cartier dual of the quotient $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, acting on $\mathrm{Spec} \mathbb{Z}[Q]$ in the natural manner.

For $n \in \mathbb{N}$, denote $\frac{1}{n}\sqrt[n]{B}$ by $\sqrt[n]{B}$. These root stacks form an inverse system: if $n \mid m$ there is a natural map $\sqrt[m]{B} \rightarrow \sqrt[n]{B}$. The inverse limit is the infinite root stack $\sqrt{\infty} B := \varprojlim_n \sqrt[n]{B}$.

2.3. The Kato-Nakayama space. Let X be a log analytic space. Its Kato-Nakayama space X_{\log} is a topological space (in good cases, a manifold with boundary), defined as follows. As a set, elements of X_{\log} are pairs (x, ϕ) consisting of a point $x \in X$ and a homomorphism of groups $\phi: M_x^{\text{gp}} \rightarrow S^1$ such that $\phi(f) = \frac{f(x)}{|f(x)|}$ for every $f \in \mathcal{O}_{X,x}^\times \subseteq M_x^{\text{gp}}$.

If $X = \mathbb{C}(P) = (\text{Spec } \mathbb{C}[P])_{\text{an}}$ for a fine monoid P , then the space X_{\log} can be identified with $\text{Hom}(P, \mathbb{R}_{\geq 0} \times S^1)$. More generally, if the log analytic space X has a Kato chart $X \rightarrow \mathbb{C}(P)$, then X_{\log} can be identified with a closed subset of the topological space $X \times \text{Hom}(P^{\text{gp}}, S^1)$ (where $\text{Hom}(P^{\text{gp}}, S^1)$ has its natural topology), and we can equip it with the induced topology. This can be shown to be independent of the particular Kato chart that we choose, and we obtain a topology on the set X_{\log} for a general X .

The natural projection $X_{\log} \rightarrow X$ that sends (x, ϕ) to x is continuous and proper. The fiber over a point $x \in X$ can be identified with the space $\text{Hom}(\overline{M}_x^{\text{gp}}, S^1)$, which is non-canonically isomorphic to a real torus $(S^1)^r$, where r is the rank of the (finitely generated) free abelian group $\overline{M}_x^{\text{gp}}$. If the log structure of X is given by a normal crossings divisor $D \subseteq X$, then the space X_{\log} is the “real oriented blowup” of X along D .

In the following we will also make use of a covering space $\tilde{X}_{\log} \rightarrow X_{\log}$, that can be constructed in presence of a Kato chart $X \rightarrow \mathbb{C}(P)$. For $\mathbb{C}(P)$ itself, this is defined as $\widetilde{\mathbb{C}(P)}_{\log} := \text{Hom}(P, \mathbb{H})$, where \mathbb{H} is the “closed complex half-plane” $\mathbb{R}_{\geq 0} \times \mathbb{R} \subseteq \mathbb{C}$ (note that usually \mathbb{H} denotes the open half-plane), and the map $\widetilde{\mathbb{C}(P)}_{\log} \rightarrow \mathbb{C}(P)_{\log} = \text{Hom}(P, \mathbb{R}_{\geq 0} \times S^1)$ is given by composing with the map $\mathbb{H} \rightarrow \mathbb{R}_{\geq 0} \times S^1$ described as $(x, y) \mapsto (x, e^{iy})$. For a general X , the map $\tilde{X}_{\log} \rightarrow X_{\log}$ is obtained by base change along the Kato chart $X \rightarrow \mathbb{C}(P)$.

In both cases, $\tilde{X}_{\log} \rightarrow X_{\log}$ is a covering space, with group of deck transformations given by $\mathbb{Z}(P) := \text{Hom}(P, \mathbb{Z})$ (or, more precisely, $\text{Hom}(P, \mathbb{Z}(1))$ where $\mathbb{Z}(1) = 2\pi i\mathbb{Z}$ - we will systematically omit these “Tate twists” in the notation). Note that this can be non-canonically identified with \mathbb{Z}^n , via $\text{Hom}(P, \mathbb{Z}) = \text{Hom}(P^{\text{gp}}, \mathbb{Z})$ and the fact that $P^{\text{gp}} \cong \mathbb{Z}^n$ for some n . This covering space should be thought of as an “atlas” of X_{\log} , the analogue of the scheme $X \times_{\text{Spec } \mathbb{Z}[P]} \text{Spec } \mathbb{Z}[Q]$ in the local description (1) for root stacks.

In fact, if $Q = \frac{1}{n}P$, and $P \rightarrow Q = \frac{1}{n}P$ is the inclusion, the group $\mu_{Q/P}$ is isomorphic to $\mu_n(P) := \text{Hom}(P, \mathbb{Z}/n\mathbb{Z}) \cong \mu_n^r$, where r is the rank of P^{gp} , and the group of deck transformations $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z})$ of the cover $\tilde{X}_{\log} \rightarrow X_{\log}$ naturally maps to $\mu_n(P)$. There is also a canonical map $\tilde{X}_{\log} \rightarrow X \times_{\mathbb{C}(P)} \mathbb{C}(\frac{1}{n}P)$ that is $(\mathbb{Z}(P) \rightarrow \mu_n(P))$ -equivariant, and this gives a canonical morphism from X_{\log} to the root stack $\sqrt[n]{X}$ (more precisely, to the underlying topological stack). If $X = \mathbb{C}(P)$, the map $\widetilde{\mathbb{C}(P)}_{\log} = \text{Hom}(P, \mathbb{H}) \rightarrow \mathbb{C}(\frac{1}{n}P) = \text{Hom}(\frac{1}{n}P, \mathbb{C})$ is given by composing $\frac{1}{n}P \cong P \rightarrow \mathbb{H}$ with $\mathbb{H} \rightarrow \mathbb{C}$ given by $(x, y) \mapsto (\sqrt[n]{x}, y/n)$ (this step “compensates” the identification $\frac{1}{n}P \cong P$), and then with $\mathbb{H} \rightarrow \mathbb{C}$ given by $(x, y) \mapsto x \cdot e^{iy}$.

The construction of this map can be globalized (see [6] and [23]), so for every fine saturated log analytic space X and every n (including $n = \infty$) there is a canonical morphism of topological stacks $\phi_n: X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$. The rough idea here is that on X_{\log} we have formal logarithms of sections of \overline{M} , so in particular we also have n -th roots of such sections, for any n , since $(\exp(\frac{1}{n}\log(z)))^n = z$.

On X_{\log} there is a sheaf of rings \mathcal{O}_X^{\log} , that is morally generated over $\tau^{-1}\mathcal{O}_X$ by formal logarithms of sections of the sheaf M . Its precise definition will be recalled later (Section 4.3).

3. PARABOLIC SHEAVES WITH REAL WEIGHTS

For this section, X will be either a fine saturated log scheme or log analytic space.

3.1. Sheaves of weights. As recalled in (2.2), to define root stacks and parabolic sheaves with finitely generated weights, one considers an injective map of Kummer type $\overline{M} \rightarrow B$ with B a coherent sheaf of monoids (i.e. it admits local charts by finitely generated monoids). The root stack $\sqrt[B]{X}$ parametrizes extensions $N: B \rightarrow \text{Div}_X$ of the DF structure of X , and parabolic sheaves are cartesian functors $E: B^{\text{wt}} \rightarrow \text{Qcoh}_X$, where B^{wt} is the “category of weights” associated with B : its objects are sections of B^{gp} , and an arrow $s \rightarrow t$ is a section b of B such that $t = s + b$ (see [5, Section 5]). Note that since $\overline{M} \rightarrow B$ is Kummer, we can see B as a subsheaf of $B_{\mathbb{Q}} \cong \overline{M}_{\mathbb{Q}}$ (for a monoid P , we denote by $P_{\mathbb{Q}}$ the positive rational cone spanned by P in $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$). In the limit, when we consider the infinite root stack $\sqrt[\infty]{X}$, the sheaf B is $\overline{M}_{\mathbb{Q}}$ itself.

Here we want to generalize these concepts to the case where we have weights in a sheaf of monoids Λ with $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$, where $\overline{M}_{\mathbb{R}} = “\overline{M} \otimes \mathbb{R}_+”$ is the positive *real* cone spanned by \overline{M} in $\overline{M}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. In other words, sections of $\overline{M}_{\mathbb{R}}$ are of the form $r \cdot m$ with $m \in \overline{M}$ and $r \in \mathbb{R}_+$. Note that Λ might well not be finitely generated as a monoid (this already happens for $\overline{M}_{\mathbb{Q}}$). The concept of *parabolic sheaves with real weights* that we will define is a generalization of earlier definitions (for example [16, 14, 9]).

Since finitely presented parabolic sheaves will be defined as the ones obtained by applying an induction functor via a “fine sub-system of weights” (see Definition 3.9 below) that does not have to be a sheaf of monoids, we will discuss sheaves of weights in a more general context, just requiring that they be sheaves of pre-ordered sets, with an action of the sheaf \overline{M}^{gp} . Part of what follows is taken from a section in a preliminary version of [5], that was removed in the final version. I am grateful to A. Vistoli and N. Borne for allowing me to include this material here.

Recall that a pre-ordered set is a set W equipped with a reflexive and transitive relation, that we denote by \leq (or \leq_W when we have to specify W). Pre-ordered sets form a category (PreOrd), where morphisms $W \rightarrow W'$ are functions $f: W \rightarrow W'$ such that $f(w) \leq_{W'} f(w')$ if $w \leq_W w'$. Pre-ordered sets can also be seen as small categories with at most one morphism (in each direction) between any two objects.

Definition 3.1. Let P be an integral monoid. A *weight system* for P is a pre-ordered set W with an action of P^{gp} , that we denote by $(p, w) \mapsto p + w$, such that if $w \leq w'$, then $p + w \leq p + w'$, and for every $p \in P$ we have $p + w \geq w$.

Let \mathcal{C} be a site, that we will later specify to be the small étale site of a scheme, or the classical site of a complex analytic space.

Definition 3.2. A *pre-sheaf of pre-ordered sets* on \mathcal{C} is a functor $W: \mathcal{C}^{\text{op}} \rightarrow (\text{PreOrd})$. A *sheaf of pre-ordered sets* on \mathcal{C} is a pre-sheaf of pre-ordered set, which is furthermore a sheaf of sets, and such that if $w, w' \in W(U)$ are such that $f_i^* w \leq f_i^* w'$ for a covering $\{f_i: U_i \rightarrow U\}$ in \mathcal{C} , then $w \leq w'$ in $W(U)$.

One can sheafify a pre-sheaf of pre-ordered sets to a sheaf of pre-ordered sets in a unique way.

Let now X be a log scheme or log analytic space, with DF structure $L: \overline{M} \rightarrow \text{Div}_X$.

Definition 3.3. A *pre-weight system* on X is a pre-sheaf of preordered sets W , together with an action of \overline{M}^{gp} , such that for every U the set $W(U)$ is a weight system for the monoid $\overline{M}(U)$. A pre-weight system is a *weight system* if W is a sheaf of pre-ordered sets.

Example 3.4. Assume that B is a sheaf of integral monoids containing \overline{M} . Then there is a natural partial order on B^{gp} , by declaring that $b \leq b'$ if and only if there exists $b'' \in B$ such that $b' = b + b''$. Moreover, there is an action of $\overline{M}^{\text{gp}} \subseteq B^{\text{gp}}$, given by the monoid operation. We will denote the corresponding weight system by B^{wt} .

If $\overline{M} \rightarrow B$ is a Kummer extension, these weight systems B^{wt} are the ones appearing in [5].

3.2. Systems of \mathcal{O} -modules. Let X be a scheme or analytic space, with a symmetric monoidal functor $L: P \rightarrow \text{Div}(X)$ for an integral monoid P . We denote the image of $p \in P$ by (L_p, s_p) . Recall that the functor L can be extended to a functor $P^{\text{gp}} \rightarrow \text{Pic}(X)$ (that we continue to denote by L), by sending $p - p'$ to the invertible sheaf $L_p \otimes_{\mathcal{O}_X} L_{p'}^\vee$ (see [5, Proposition 5.2]). Denote by $\lambda_{a,b}: L_{a+b} \cong L_a \otimes_{\mathcal{O}_X} L_b$ and $\epsilon: L_0 \cong \mathcal{O}_X$ the isomorphisms that are part of the data of the symmetric monoidal functor L (as in [5, Definition 2.1]).

Let W be a weight system for the monoid P . The following is the straightforward adaptation of [5, Definition 5.6] to this more general setting.

Definition 3.5. A *system of \mathcal{O} -modules* (E, j^E) for the above data is a functor $E: W \rightarrow \text{Mod}(\mathcal{O}_X)$, denoted $w \mapsto E_w$ and $(w \leq w') \mapsto E_{(w,w')}$, together with an isomorphism

$$\rho_{a,w}^E: E_{a+w} \cong L_a \otimes_{\mathcal{O}_X} E_w$$

for every $a \in P^{\text{gp}}$ and $w \in W$, such that

- (a) for every $p \in P$ the diagram

$$\begin{array}{ccc} E_w & \xrightarrow{E_{(w,p+w)}} & E_{p+w} \\ \cong \downarrow & & \downarrow \rho_{p,w}^E \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} E_w & \xrightarrow{s_p \otimes \text{id}} & L_p \otimes_{\mathcal{O}_X} E_w \end{array}$$

commutes,

- (b) for every $w \leq w'$ in W and $a \in P^{\text{gp}}$, then the diagram

$$\begin{array}{ccc} E_{a+w} & \xrightarrow{\rho_{a,w}^E} & L_a \otimes_{\mathcal{O}_X} E_w \\ E_{(a+w,a+w')} \downarrow & & \downarrow \text{id} \otimes E_{(w,w')} \\ E_{a+w'} & \xrightarrow{\rho_{a,w'}^E} & L_a \otimes_{\mathcal{O}_X} E_{w'} \end{array}$$

commutes,

- (c) for every $a, b \in P^{\text{gp}}$ and $w \in W$, the diagram

$$\begin{array}{ccc} E_{a+b+w} & \xrightarrow{\rho_{a+b,w}^E} & L_{a+b} \otimes_{\mathcal{O}_X} E_w \\ \rho_{a,b+w}^E \downarrow & & \downarrow \lambda_{a,b} \otimes \text{id} \\ L_a \otimes_{\mathcal{O}_X} E_{b+w} & \xrightarrow{\text{id} \otimes \rho_{b,w}^E} & L_a \otimes_{\mathcal{O}_X} L_b \otimes_{\mathcal{O}_X} E_w \end{array}$$

commutes, and

- (d) for every $w \in W$ the composite

$$E_w = E_{0+w} \xrightarrow{\rho_{0,w}^E} L_0 \otimes_{\mathcal{O}_X} E_w \xrightarrow{\epsilon \otimes \text{id}} \mathcal{O}_X \otimes_{\mathcal{O}_X} E_w$$

coincides with the natural isomorphism $E_w \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} E_w$.

A morphism of systems of \mathcal{O} -modules is a natural transformation $\phi: E \rightarrow E'$, such that for every $a \in P^{\text{gp}}$ and $w \in W$, the diagram

$$\begin{array}{ccc} E_{a+w} & \xrightarrow{\rho_{a,w}^E} & L_a \otimes_{\mathcal{O}_X} E_w \\ \phi_{a+w} \downarrow & & \downarrow \text{id} \otimes \phi_w \\ E'_{a+w} & \xrightarrow{\rho_{a,w}^{E'}} & L_a \otimes_{\mathcal{O}_X} E'_w \end{array}$$

commutes. Systems of \mathcal{O} -modules on X with respect to W form an abelian category $\text{Mod}(X, W)$. The abelian structure is defined “component-wise”.

We say that a system of \mathcal{O} -modules E is *quasi-coherent* if the functor E has values in $\text{Qcoh}(X)$. We denote by $\text{Qcoh}(X, W)$ the full sub-category of $\text{Mod}(X, W)$ of quasi-coherent systems of \mathcal{O} -modules.

Assume now that X is a log scheme or log analytic space, with DF structure given by $L: \overline{M} \rightarrow \text{Div}_X$, and assume that we have a weight system W for \overline{M} on X .

Definition 3.6. A *system of \mathcal{O} -modules* for the above data is a cartesian functor $E: W \rightarrow \text{Mod}_{\mathcal{O}_X}$, with an isomorphism of \mathcal{O}_U -modules

$$\rho_{a,w}^E: E_{a+w} \cong L_a \otimes_{\mathcal{O}_U} E_w$$

for every open $U \rightarrow X$ (either an étale morphism, or an open immersion of analytic spaces), $a \in \overline{M}^{\text{gp}}(U)$ and $w \in W(U)$, such that

- a) for every open $U \rightarrow X$, the restriction $E(U): W(U) \rightarrow \text{Mod}(\mathcal{O}_U)$ is a system of \mathcal{O} -modules on U with weights in $W(U)$, and
- b) for every arrow $f: (U \rightarrow X) \rightarrow (V \rightarrow X)$ between opens of X , and for every $a \in \overline{M}^{\text{gp}}(V)$ and $w \in W(V)$, the isomorphism

$$\rho_{f^*a, f^*w}^E: E_{f^*(a+w)} = E_{f^*a + f^*w} \cong L_{f^*a} \otimes_{\mathcal{O}_U} E_{f^*w}$$

coincides with the pullback of $\rho_{a,w}^E: E_{a+w} \cong L_a \otimes_{\mathcal{O}_V} E_w$.

Sometimes we will refer to the sheaves E_w as *pieces* of the system of \mathcal{O} -modules E . Similarly to the previous case, there is an abelian category $\text{Mod}(X, W)$ of systems of \mathcal{O}_X -modules on X with weights in W , and a full subcategory $\text{Qcoh}(X, W) \subseteq \text{Mod}(X, W)$ of quasi-coherent systems.

These definitions coincide with to the ones of [5, Section 5.2], if the weight system is given by a Kummer extension of sheaves of monoids $\overline{M} \rightarrow B$.

3.2.1. Functoriality. Let X be a log scheme and W, W' two weight systems, with an injective \overline{M}^{gp} -equivariant map $j: W \rightarrow W'$. We will call such a map an *embedding* of weight systems.

In this situation, we can define two adjoint functors between systems of \mathcal{O} -modules, that we call *restriction*

$$\text{Res}_W^{W'}: \text{Mod}(X, W') \rightarrow \text{Mod}(X, W)$$

and *induction*

$$\text{Ind}_W^{W'}: \text{Mod}(X, W) \rightarrow \text{Mod}(X, W').$$

Restriction is simply defined by restricting a system of \mathcal{O} -modules $E: W' \rightarrow \text{Mod}_{\mathcal{O}_X}$ and the isomorphisms ρ^E along the embedding $W \rightarrow W'$. Note that this operation sends quasi-coherent systems to quasi-coherent systems.

Induction is more complicated to describe. Assume that $E: W \rightarrow \text{Mod}_{\mathcal{O}_X}$ is a system of \mathcal{O} -modules, and let $U \rightarrow X$ be an open, and $w' \in W'(U)$. We want to define a sheaf of \mathcal{O}_U -modules $\tilde{E}_{w'}$.

For an open $f: V \rightarrow U$, consider the subset of $W(U)$ given by

$$W_{w'}(V) = \{w \in W(V) \mid w \leq_{W'} f^* w'\}$$

with the induced pre-order. We have a functor from $W_{w'}(V)$ to the category of abelian groups, sending w to $E_w(V)$. Define

$$\tilde{E}_{w'}^{\text{pre}}(V) = \varinjlim_{w \in W_{w'}(V)} E_w(V).$$

Moreover, given a further open $g: V' \rightarrow V$, we have a morphism of pre-ordered sets $g^*: W_{w'}(V) \rightarrow W_{w'}(V')$. For every $w \in W_{w'}(V)$ we have an isomorphism $g^* E_w \cong E_{g^* w}$, and these induce a homomorphism

$$E_w(V) \rightarrow E_{h^* w}(V') \rightarrow \varinjlim_{w'' \in W_{w'}(V')} E_{w''}(V') = \tilde{E}_{w'}^{\text{pre}}(V').$$

By taking the colimit we obtain a homomorphism $\tilde{E}_{w'}^{\text{pre}}(V) \rightarrow \tilde{E}_{w'}^{\text{pre}}(V')$. This makes $\tilde{E}_{w'}^{\text{pre}}$ into a presheaf of \mathcal{O}_U -modules. Let $\tilde{E}_{w'}$ be the associated sheaf.

We define a cartesian functor $\tilde{E}: W' \rightarrow \text{Mod}_{\mathcal{O}_X}$ sending w' to $\tilde{E}_{w'}$. Note that if $w' \leq w''$ in W' , then there is an inclusion of pre-ordered sets $W_{w'}(V) \rightarrow W_{w''}(V)$ for every open $V \rightarrow U$, and this induces a homomorphism of \mathcal{O}_U -modules $E_{w'} \rightarrow E_{w''}$. Since pullbacks respect direct limit and sheafifications, it also follows that the functor is cartesian. Moreover the isomorphisms $\rho_{a,w}^E$ induce isomorphisms $\rho_{a,w'}^{\tilde{E}}: \tilde{E}_{a+w'} \cong L_a \otimes_{\mathcal{O}_X} \tilde{E}_{w'}$ by taking colimits.

It is straightforward now to define the functor $\text{Ind}_W^{W'}$ sending E to \tilde{E} . Moreover, one also easily checks that $\text{Ind}_W^{W'}$ is left adjoint to $\text{Res}_W^{W'}$, and fully faithful (equivalently, the unit of the adjunction $\text{id} \rightarrow \text{Res}_W^{W'} \circ \text{Ind}_W^{W'}$ is an isomorphism). We record this in the following proposition.

Proposition 3.7. *The restriction functor $\text{Res}_W^{W'}$ has a right adjoint $\text{Ind}_W^{W'}$, which is moreover fully faithful.* \square

There are obvious (simpler) versions of these constructions for systems of \mathcal{O} -modules for a weight system relative to a monoid P and a symmetric monoidal functor $L: P \rightarrow \text{Div}(X)$.

Using the equivalence between parabolic sheaves and quasi-coherent sheaves on root stacks of [5], these two functors are identified with pullback and pushforward along the canonical map between the two corresponding root stacks, and this adjunction is the usual one. This is explained for example in [21, Section 2.2] and [22, Section 7.1].

Remark 3.8. While $\text{Res}_W^{W'}$ always preserves quasi-coherence, the functor $\text{Ind}_W^{W'}$ probably does not, in full generality. We will shortly show that it indeed does, when the weight system W has discrete local models.

3.2.2. Local models. We now discuss charts for weight systems. Assume that X is a log scheme or a log analytic space, with DF structure $L: \overline{M} \rightarrow \text{Div}_X$ with a global chart $P \rightarrow \overline{M}(X)$, and that R is a weight system for P . Then we obtain an induced weight system W for \overline{M} as follows. Call $K \subseteq P_X$ the kernel of the map to \overline{M} , and consider the (sheaf) quotient $W = R_X/K$. This is a weight system for \overline{M} , for the pre-order defined by $w \leq w'$ in $W(U)$ if there exists a covering $\{f_i: U_i \rightarrow U\}$ and $w_i, w'_i \in R_X(U_i)$ such that $w_i \leq w'_i$ for every i , and $f_i^* w = w_i$, $f_i^* w' = w'_i$ for every i . There is a natural map $R \rightarrow W(U)$ of pre-ordered sets.

Definition 3.9. In the situation we just described, we say that the pair $(P \rightarrow \overline{M}(X), R \rightarrow W(X))$ is a *chart* for the weight system W on X .

A chart is said to be *fine* if P is finitely generated, and R is the union of a finite number of orbits for the action of P^{gp} .

Sometimes we will refer to a chart for a weight system just via the morphism $R \rightarrow W(X)$.

Example 3.10. Consider the weight system associated with the monoid $\frac{1}{n}\overline{M}$. If the DF structure has a global chart $P \rightarrow \overline{M}(X)$, this weight system also has a global chart, given by $\frac{1}{n}P \rightarrow \frac{1}{n}\overline{M}(X)$. This chart is also fine, since the pre-ordered set $\frac{1}{n}P^{\text{wt}}$ has finitely many orbits with respect to the action of P^{gp} .

Analogously the weight system associated with $\overline{M}_{\mathbb{Q}}$ has a chart given by $P_{\mathbb{Q}} \rightarrow \overline{M}_{\mathbb{Q}}(X)$, but this chart is not fine.

Definition 3.11. A weight system W for a log scheme X is said to be *quasi-coherent* if it locally admits charts. It is said to be *fine* if it locally admits fine charts.

Proposition 3.12. *With the notation of the discussion preceding Proposition 3.7, if the system W is fine, then the induction functor*

$$\text{Ind}_W^{W'} : \text{Mod}(X, W) \rightarrow \text{Mod}(X, W')$$

sends quasi-coherent systems to quasi-coherent systems.

Proof. This can be checked locally, where it follows from the fact that a colimit of a finite diagram of quasi-coherent sheaves is quasi-coherent. \square

Remark 3.13. If X is a scheme, then a filtered colimit of quasi-coherent sheaves is quasi-coherent [20, Tag 0781], and in the above Proposition we can only require that W be quasi-coherent. In the analytic context, the preceding assertion may fail (see [7, Remark 2.1.5]).

3.3. Parabolic sheaves. Assume now that Λ is a sheaf of monoids on the fine saturated logarithmic scheme X , such that $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$, and that it is *quasi-coherent*, i.e. it admits local charts. This means that locally on X there is a chart $P \rightarrow \overline{M}(X)$ and a monoid Λ_0 with $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$, with a chart $\Lambda_0 \rightarrow \Lambda(X)$ that makes the obvious diagram commute. One can check that this is equivalent to asking that Λ be *log constructible* [18, 3.2] (briefly, this means that it is locally constant on the stratification associated to the log structure of X).

We will also assume that $\Lambda \subseteq \overline{M}_{\mathbb{R}}$ is *saturated*, meaning that $\Lambda = \Lambda^{\text{gp}} \cap \overline{M}_{\mathbb{R}}$. This implies that if $\lambda \geq \mu$ in $\overline{M}_{\mathbb{R}}$, then $\lambda - \mu$ is actually a section of Λ . This property can also be checked on local models $\Lambda_0 \subseteq P_{\mathbb{R}}$. Note that if such a Λ is finitely generated, then it is necessarily contained in some $\frac{1}{n}\overline{M}$.

Definition 3.14. A *parabolic sheaf* on X with weights in Λ is a system of \mathcal{O} -modules for the weight system Λ^{wt} .

The same definition applies in the presence of a Kato chart $P \rightarrow \overline{M}(X)$ and a monoid $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$, and gives a notion of parabolic sheaf on X with weights in Λ_0 .

As for the case of schemes and finite systems of weights treated in [5], we want to restrict to a class of “quasi-coherent” parabolic sheaves. One natural choice would be to consider quasi-coherent systems of \mathcal{O} -modules in the definition above, but we will do something a little different. Let us define finitely presented sheaves first.

Definition 3.15. A parabolic sheaf E with weights in Λ is *finitely presented* if for all $\lambda \in \Lambda^{\text{wt}}$ the sheaf E_λ is a finitely presented sheaf of \mathcal{O}_X -modules on X , and étale locally on X there exists a fine sub-weight system $R \subseteq \Lambda^{\text{wt}}$ such that E is in the essential image of the induction functor

$$\text{Ind}_R^{\Lambda^{\text{wt}}} : \text{Mod}(X, R) \rightarrow \text{Mod}(X, \Lambda^{\text{wt}}).$$

Intuitively, the last condition says that locally the parabolic sheaf is completely determined by finitely many of its pieces E_λ .

Example 3.16. Assume that the log structure has a global chart with monoid \mathbb{N} (so the DF structure is given by a line bundle with a section $(L, s) \in \text{Div}(X)$), and that $\Lambda = \mathbb{R}$. In this case a parabolic sheaf with weights in Λ is the assignment of an \mathcal{O}_X -module E_r for each $r \in \mathbb{R}$, with maps $E_r \rightarrow E_{r'}$ when $r \geq r'$, that are compatible with respect to composition, and such that $E_r \rightarrow E_{r+1} \cong L \otimes_{\mathcal{O}_X} E_r$ is identified with multiplication by the section s .

For such a sheaf, being finitely presented means that each E_r is a finitely presented sheaf on X , and moreover there exist finitely many real numbers $0 \leq r_1 < \dots < r_k < 1$, such that for every $r \in \mathbb{R}$ the sheaf E_r is obtained in this way: consider the largest integer $[r]$ which is $\leq r$, and the fractional part $s = \{r\} = r - [r]$; then $E_r = E_{r_i} \otimes L_{[r]}$, where r_i is the largest of the fixed numbers that is $\leq s$. The maps $E_r \rightarrow E_s$ are obtained in the evident manner.

In other words, the parabolic sheaf E is completely determined by the weights r_i , the finitely presented sheaves E_{r_i} , and the maps between them.

Remark 3.17. Note that if X is a noetherian (or more generally coherent) scheme or an analytic space, finitely presented sheaves coincide with coherent sheaves. In the next section we will exclusively work with complex analytic spaces, so this comment will apply.

In the category of parabolic sheaves with weights in Λ , we can form colimits by taking the colimits “level-wise”.

Definition 3.18. A parabolic sheaf with weights in Λ is *quasi-coherent* if locally on X it can be written as a filtered colimit of finitely presented parabolic sheaves with weights in Λ .

Remark 3.19. This definition is inspired by the definition of a quasi-coherent sheaf on an analytic space of [7]. We opted to use this notion, instead of the perhaps more natural one requiring that all sheaves E_λ are quasi-coherent on X , for technical convenience. Note that, if X itself is coherent, for a quasi-coherent sheaf in the sense of the definition it is indeed the case that E_λ is quasi-coherent for every λ (it is locally a filtered colimit of coherent sheaves), but it is not clear if the two notions would coincide in general.

See also the discussion about quasi-coherent sheaves on X_{\log} in (4.5).

We will denote by $\text{Par}(X, \Lambda)$ the category of quasi-coherent parabolic sheaves on X with weights in Λ , and by $\text{FPPar}(X, \Lambda)$ the full subcategory of finitely presented parabolic sheaves. Note that it is not clear that these are abelian categories, but we can talk about exactness by embedding these categories into the abelian category $\text{Mod}(X, \Lambda^{\text{wt}})$ of systems of \mathcal{O} -modules for Λ^{wt} .

We will denote by $\text{Par}(X, \mathbb{Q})$ the category $\text{Par}(X, \overline{M}_{\mathbb{Q}})$, and by $\text{Par}(X, \mathbb{R})$ the category $\text{Par}(X, \overline{M}_{\mathbb{R}})$.

Proposition 3.20. *Let X be a fine saturated log scheme, and Λ be a sheaf of monoids with $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$, with a global chart $(P \rightarrow \overline{M}(X), \Lambda_0 \rightarrow \Lambda(X))$. Then there is an equivalence of categories $\iota : \text{Par}(X, \Lambda) \rightarrow \text{Par}(X, \Lambda_0)$.*

Proof. The functor ι is defined by restricting $E : \Lambda^{\text{wt}} \rightarrow \text{Qcoh}_X$ and the isomorphisms $\rho_{a,w}^E$ along $\Lambda_0^{\text{wt}} \rightarrow \Lambda^{\text{wt}}(X)$.

The fact that ι is an equivalence can be proven exactly as in [5, Proposition 5.10] (the only change is that in [5, Lemma 5.11] one has to consider a section l such that $k \leq l \leq mk$ for some $m \in \mathbb{N}$). \square

To conclude this section we briefly note that, over the complex numbers, there is a version for finitely presented parabolic sheaves of the GAGA equivalence, that relates parabolic sheaves on a proper scheme X and parabolic sheaves on the associated analytic space. Assume that X is a fine saturated log scheme locally of finite type over the complex numbers. Then the analytification X_{an} inherits a fine saturated log structure (on its classical site), and we can compare parabolic sheaves on the two sides. Let Λ be a sheaf of monoids on X such that $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$, and denote by Λ_{an} the induced sheaf on the classical site of X_{an} .

Proposition 3.21. *There is a natural analytification functor*

$$(-)_{\text{an}}: \text{FPPar}(X, \Lambda) \rightarrow \text{FPPar}(X_{\text{an}}, \Lambda_{\text{an}})$$

which is exact. If X is proper, this functor is an equivalence of categories.

Proof. Note that since X is noetherian, the pieces of a finitely presented parabolic sheaf are coherent sheaves. The analytification functor is then defined by analytifying all the pieces and the maps of a parabolic sheaf, and all the assertions follow immediately from the classical GAGA theorems. \square

3.4. Correspondence with sheaves on root stacks. Since our proof of Theorem 5.1 will follow quite closely the one of [5, Theorem 6.1], we give a short reminder about that case.

Let X be a fine saturated log scheme, and consider a system of denominators $\overline{M} \rightarrow B$ (i.e. a Kummer extension of sheaves of monoids, admitting local charts). We are going sketch the construction of the functor $\Phi: \text{Qcoh}(\sqrt[B]{X}) \rightarrow \text{Par}(X, B)$, and the proof that it is an equivalence.

Recall that $\pi: \sqrt[B]{X} \rightarrow X$ carries a universal DF structure $N: \pi^{-1}B \rightarrow \text{Div}_{\sqrt[B]{X}}$ that extends the pullback $\pi^*L: \pi^{-1}A \rightarrow \text{Div}_{\sqrt[B]{X}}$. Given a quasi-coherent sheaf $F \in \text{Qcoh}(\sqrt[B]{X})$ and $b \in B^{\text{wt}}(U)$ for $U \rightarrow X$ étale, set

$$\Phi(F)_b := \pi_*(F \otimes_{\mathcal{O}_{\sqrt[B]{X}}} N_b).$$

Note that for $b \leq b'$, i.e. $b' = b + b''$ with $b'' \in B(U)$, we have $N_{b'} \cong N_b \otimes_{\mathcal{O}_X} N_{b''}$, and hence we obtain a map $\Phi(F)_b \rightarrow \Phi(F)_{b'}$, given by multiplication by the section $s_{b''}$ of $N_{b''}$. The projection formula for π provides the isomorphisms $\rho_{m,b}^{\Phi(F)}: \Phi(F)_{b+m} \cong L_m \otimes_{\mathcal{O}_X} \Phi(F)_b$ for $b \in B^{\text{wt}}(U)$ and $m \in \overline{M}(U)$. Easy verifications show that $\Phi(F)$ is a parabolic sheaf on X with weights in B .

To prove that this functor is an equivalence, since both categories extend to stacks for the étale topology of X , one can localize where there is a Kato chart $X \rightarrow \text{Spec } \mathbb{Z}[P]$ and a chart $(Q \rightarrow B(X), P \rightarrow Q)$ for $\overline{M} \rightarrow B$, and construct a quasi-inverse locally. Recall from (2.2) that in the presence of such charts, quasi-coherent sheaves on $\sqrt[B]{X}$ can be identified with quasi-coherent sheaves of $\mathcal{O}_X[P^{\text{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ -modules on X that have a Q^{gp} -grading compatible with the module structure.

Given a parabolic sheaf E , one forms the sheaf $\bigoplus_{q \in Q^{\text{gp}}} E_q$ on X . This has a structure of $\mathcal{O}_X[P^{\text{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ -module, determined by the maps $E_q \rightarrow E_{q'}$ for $q \leq q'$ that are part of the definition of a parabolic sheaf, and a Q^{gp} -grading that is compatible with the module structure. Hence we obtain a quasi-coherent sheaf on $\sqrt[B]{X}$, and this gives the desired quasi-inverse. One can also show that if X is noetherian, coherent sheaves on $\sqrt[B]{X}$ correspond to parabolic sheaves such that each E_b is coherent.

4. SHEAVES ON THE KATO-NAKAYAMA SPACE

From now on X will be a fine saturated complex analytic space (which might for example be the analytification of a fine saturated log scheme locally of finite type over \mathbb{C}). We will denote by $\tau: X_{\log} \rightarrow X$ the Kato-Nakayama space of X with its natural projection.

4.1. Indexed algebras. Assume for the moment that (T, \mathcal{O}_T) is an arbitrary ringed space.

We start by describing a construction of a sheaf of \mathcal{O}_T -algebras \mathcal{A} associated with an extension

$$(2) \quad 0 \longrightarrow \mathcal{O}_T^\times \longrightarrow M \xrightarrow{\pi} \overline{M} \longrightarrow 0$$

of sheaves of monoids on T , which could be associated with a log structure in the case where T is the underlying space of a complex analytic space (but we will also apply this to exact sequences on the Kato-Nakayama space). By an extension of sheaves of monoids, we mean a pair of maps $f: P' \rightarrow P$ and $g: P \rightarrow P''$, such that f is an isomorphism onto the submonoid $g^{-1}(0)$ of P , and g induces an isomorphism $P/P' \cong P/f^{-1}(0) \cong P''$. The following construction is taken from a paper of Lorenzon [13], via the work of Ogus [18].

For an open $U \subseteq T$ and a section $a \in \overline{M}(U)$, the sheaf of preimages of a in M is an \mathcal{O}_T^\times -torsor that we denote by P_a . This corresponds to a line bundle N_a (i.e. an invertible sheaf of \mathcal{O}_T -modules) on T , and we define $\mathcal{A}(U) = \bigoplus_{a \in \overline{M}(U)} N_a(U)$ as an $\mathcal{O}_T(U)$ -module. Natural restriction maps give a sheaf of \mathcal{O}_T -modules \mathcal{A} . To ease notation, we will succinctly write $\mathcal{A} = \bigoplus_{a \in \overline{M}} N_a$. Moreover if $a, b \in \overline{M}(U)$, we have a natural map $N_a \otimes_{\mathcal{O}_T} N_b \rightarrow N_{a+b}$, and this gives \mathcal{A} the structure of a sheaf of \mathcal{O}_T -algebras. There is also a natural morphism of monoids $M \rightarrow \mathcal{A}$ (where the operation on \mathcal{A} is multiplication).

Assume now that we have a homomorphism of sheaves of monoids $M \rightarrow \mathcal{O}_T$ (where \mathcal{O}_T is equipped with multiplication). Then every N_a has an induced co-section $t_a: N_a \rightarrow \mathcal{O}_T$, induced by the natural map $P_a \subseteq M \rightarrow \mathcal{O}_T$.

Example 4.1. Assume we are considering the extension (2) associated with the natural log structure given by the origin on $X = \mathbb{A}^1 = (\text{Spec } \mathbb{C}[z])_{\text{an}}$. In this case the sheaf \mathcal{A} is as follows. If $U \subseteq \mathbb{A}^1$ does not contain the origin, then $\mathcal{A}|_U \cong \mathcal{O}_U$, since in this case the restriction of the extension is trivial (i.e. $\overline{M} = 0$). If U does contain the origin, then $\Gamma(U, \overline{M}) = \mathbb{N}$, and $\mathcal{A}(U) \cong \bigoplus_{n \in \mathbb{N}} t^n \cdot \mathcal{O}_X(U)$, where t^n is just a placeholder variable.

The map $N_a \otimes_{\mathcal{O}_T} N_b \rightarrow N_{a+b}$ is given by the natural isomorphism

$$(t^n \cdot \mathcal{O}_{\mathbb{A}^1}) \otimes_{\mathcal{O}_{\mathbb{A}^1}} (t^m \cdot \mathcal{O}_{\mathbb{A}^1}) \cong t^{n+m} \cdot \mathcal{O}_{\mathbb{A}^1},$$

and the co-section $N_a \rightarrow \mathcal{O}_X$ is determined by $t^n \cdot \mathcal{O}_{\mathbb{A}^1} \rightarrow \mathcal{O}_{\mathbb{A}^1}$ sending t^n to z^n . A similar description can be given for affine toric varieties $X = \mathbb{C}(P) = (\text{Spec } \mathbb{C}[P])_{\text{an}}$ with the natural log structure.

In particular note that the sheaf \mathcal{A} is not quasi-coherent, even in the algebraic case (and this is in fact typical). If it were quasi-coherent, it would be the sheaf associated to the $\mathbb{C}[z]$ -module $\Gamma(\mathbb{A}^1, \mathcal{A}) = \bigoplus_{n \in \mathbb{N}} t^n \cdot \mathbb{C}[z]$, but this is clearly incorrect, since the restriction of \mathcal{A} to $\mathbb{A}^1 \setminus \{0\}$ coincides with the restriction of $\mathcal{O}_{\mathbb{A}^1}$.

Denote by $\text{Div}_{(T, \mathcal{O}_T)}$ the symmetric monoidal category over T of pairs (L, s) consisting of an \mathcal{O}_T -line bundle L (i.e. a locally free sheaf of \mathcal{O}_T -modules of rank 1) with a section s . From the extension (2) and the previous construction we also obtain a symmetric monoidal functor $\overline{M} \rightarrow \text{Div}_{(T, \mathcal{O}_T)}$ by sending $a \in \overline{M}(U)$ to the dual $L_a = N_a^\vee$ of the line bundle N_a associated to the torsor P_a , together with the section $s_a: \mathcal{O}_T \rightarrow L_a$ induced by the co-section t_a .

Definition 4.2. If X is a log analytic space, and extension (2) comes from the log structure, we will denote the associated sheaf of algebras by \mathcal{A}_X .

4.2. Extensions on X_{\log} . Let X be a fine saturated log analytic space. We will explain how to produce various extensions of the form (2) on the Kato-Nakayama space X_{\log} , besides the one coming from the log structure of X . We will use these extensions to produce sheaves of rings \mathcal{O}_Λ on X_{\log} for a quasi-coherent sheaf of submonoids $\Lambda \subseteq \overline{M}_{\mathbb{R}} = \overline{M} \otimes \mathbb{R}_+$ containing \overline{M} . Morally, the sheaf \mathcal{O}_Λ will be generated by the pullback of \mathcal{O}_X and powers m^α where m is a section of \overline{M} and $\alpha \in \mathbb{R}_+$ is such that $m \otimes \alpha \in \Lambda \subseteq \overline{M}_{\mathbb{R}}$ (see the description in (4.4)).

Recall that the universal object parametrized by the topological space X_{\log} with the projection $\tau: X_{\log} \rightarrow X$ is a homomorphism $c: \tau^{-1}M^{\text{gp}} \rightarrow S^1_{X_{\log}}$ of sheaves of abelian groups of X_{\log} , such that $c(f) = f/|f|$ for $f \in \tau^{-1}\mathcal{O}^\times$. Here and in what follows, if T is a topological space and G is a topological monoid, or group, etc. we denote by G_T the sheaf of continuous functions towards G on opens of T (with the induced structure of a sheaf of monoids, groups, etc.).

Consider the sheaf of abelian groups $\mathcal{L} = \tau^{-1}M^{\text{gp}} \times_{S^1_{X_{\log}}} i\mathbb{R}_{X_{\log}}$ on X_{\log} , where $i\mathbb{R}_{X_{\log}} \rightarrow S^1_{X_{\log}}$ is given by the exponential. This sits in a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & \mathcal{L} & \longrightarrow & \tau^{-1}M^{\text{gp}} \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow c \\ 0 & \longrightarrow & 2\pi i\mathbb{Z} & \longrightarrow & i\mathbb{R}_{X_{\log}} & \longrightarrow & S^1_{X_{\log}} \longrightarrow 0. \end{array}$$

In other words, sections of \mathcal{L} over an open U are pairs consisting of a section m of $\tau^{-1}M^{\text{gp}}$ and a continuous function $\theta: U \rightarrow i\mathbb{R}$, such that $c(m) = e^{i\theta}$ as functions $U \rightarrow S^1$. If we think of c as assigning a *phase* to every section of $\tau^{-1}M^{\text{gp}}$ that is not in $\tau^{-1}\mathcal{O}_X^\times$, then \mathcal{L} records also the choice of an *angle*, i.e. a pre-image in $i\mathbb{R}$ of the phase. In this sense, \mathcal{L} is a sheaf of “logarithms” of sections of M^{gp} . The structure sheaf \mathcal{O}_X^{\log} of X_{\log} is constructed by formally adjoining to \mathcal{O}_X the sections of the sheaf \mathcal{L} (see (4.3) for details).

Example 4.3. The sheaf \mathcal{L} on the Kato-Nakayama space S^1 of the standard log point ($\text{Spec } \mathbb{C}, \mathbb{N}$) can be described as follows: on the universal cover $\pi: \mathbb{R} \rightarrow S^1$, consider the constant sheaf $\mathbb{Z} \oplus \mathbb{C}_{\mathbb{R}}$, and make the group of deck transformations \mathbb{Z} act on this sheaf by $k \cdot (k', c) = (k', c + 2\pi k k' i)$ (so that the sheaf acquires an equivariant structure). The result of descent to S^1 is precisely \mathcal{L} . The map $2\pi i\mathbb{Z} \rightarrow \pi^{-1}\mathcal{L}$ can be described as $2\pi i k \mapsto (0, 2\pi i k)$, and $\pi^{-1}\mathcal{L} \rightarrow \pi^{-1}\tau^{-1}M^{\text{gp}} \cong \mathbb{Z} \oplus \mathbb{C}_{\mathbb{R}}^\times$ is given by $(k, c) \mapsto (k, \exp(c))$.

A little diagram chasing shows that we have an exact sequence

$$(3) \quad 0 \longrightarrow \tau^{-1}\mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow \tau^{-1}\overline{M}^{\text{gp}} \longrightarrow 0$$

on X_{\log} . The following construction is taken from [18, Section 3.3]. Let us tensor (3) by the constant sheaf \mathbb{C} (over \mathbb{Z} - we will omit this from the notation). We obtain

$$0 \longrightarrow \tau^{-1}\mathcal{O}_X \otimes \mathbb{C} \longrightarrow \mathcal{L} \otimes \mathbb{C} \longrightarrow \tau^{-1}\overline{M}^{\text{gp}} \otimes \mathbb{C} \longrightarrow 0.$$

Now let us consider the map $\tau^{-1}\mathcal{O}_X \otimes \mathbb{C} \rightarrow \tau^{-1}\mathcal{O}_X^\times$ defined on generators as $f \otimes c \mapsto e^{c \cdot f}$, and the induced diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau^{-1}\mathcal{O}_X \otimes \mathbb{C} & \longrightarrow & \mathcal{L} \otimes \mathbb{C} & \longrightarrow & \tau^{-1}\overline{M}^{\text{gp}} \otimes \mathbb{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & \tau^{-1}\mathcal{O}_X^\times & \longrightarrow & M_{\log}^{\text{gp}} & \longrightarrow & \tau^{-1}\overline{M}^{\text{gp}} \otimes \mathbb{C} \longrightarrow 0 \end{array}$$

where M_{\log}^{gp} is the pushout of the diagram to its left.

Finally, given a quasi-coherent sheaf of monoids Λ on X , with $\Lambda \subseteq \overline{M}_{\mathbb{R}} = \overline{M} \otimes \mathbb{R}_+ \subseteq \overline{M}^{\text{gp}} \otimes \mathbb{C}$, we can pullback the bottom extension of the last diagram to an extension of sheaves of monoids on X_{\log}

$$(4) \quad 0 \longrightarrow \tau^{-1}\mathcal{O}_X^{\times} \longrightarrow M_{\Lambda} \longrightarrow \tau^{-1}\Lambda \longrightarrow 0.$$

Remark 4.4. In [18], the symbol Λ is used in this same context to denote a log constructible sheaf of abelian groups in $\overline{M} \otimes \mathbb{C}$, and not a sheaf of monoids.

Definition 4.5. We will denote by \mathcal{A}_{Λ} the sheaf of $\tau^{-1}\mathcal{O}_X$ -algebras on X_{\log} , associated with the extension (4), by the procedure outlined in (4.1).

It is clear that the construction of these extensions, as well as the objects that we are going to describe next, are compatible with strict base change.

4.3. Sheaves of rings on X_{\log} . Now consider $\Lambda = \overline{M}$, so that $\mathcal{A}_{\Lambda} = \tau^{-1}\mathcal{A}_X$, and note that there is a natural homomorphism of $\tau^{-1}\mathcal{O}_X$ -algebras $\tau^{-1}\mathcal{A}_X \rightarrow \tau^{-1}\mathcal{O}_X$, which is the pullback of a homomorphism $\mathcal{A}_X \rightarrow \mathcal{O}_X$ on X : each homogeneous piece N_a of \mathcal{A}_X has a morphism of \mathcal{O}_X -modules into \mathcal{O}_X given by the co-section $s_a: N_a \rightarrow \mathcal{O}_X$, and the resulting map $\mathcal{A}_X \rightarrow \mathcal{O}_X$ is a homomorphism of algebras.

Moreover for every Λ we have a natural homomorphism $\tau^{-1}\mathcal{A}_X \rightarrow \mathcal{A}_{\Lambda}$. We set

$$\mathcal{O}_{\Lambda} := \mathcal{A}_{\Lambda} \otimes_{\tau^{-1}\mathcal{A}_X} \tau^{-1}\mathcal{O}_X.$$

This is a sheaf of rings on X_{\log} , together with an injective map $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{\Lambda}$.

As anticipated above, \mathcal{O}_{Λ} should be loosely thought of as $\tau^{-1}\mathcal{O}_X[m_i^{\alpha_i}]$, where $m_i \otimes \alpha_i$ are the sections of $\Lambda \subseteq \overline{M} \otimes \mathbb{R}_+$, and the obvious relations are satisfied (for example, m_i^1 is identified with a corresponding local section $f_i \in \tau^{-1}\mathcal{O}_X$). Note that this will often be non-finitely generated over $\tau^{-1}\mathcal{O}_X$.

See Remark 4.13 below for a more precise statement, in a particular case.

Example 4.6. Assume that X is the standard log point $(\text{Spec } \mathbb{C}, \mathbb{N})$, and let $\mathbb{N} \subseteq \Lambda \subseteq \mathbb{R}_+$ be a monoid. The algebra \mathcal{A}_X in this case can be described as the \mathbb{C} -algebra $\bigoplus_{n \in \mathbb{N}} t^n \cdot \mathbb{C}$. The morphism $\mathcal{A}_X \rightarrow \mathcal{O}_X = \mathbb{C}$ sends t^0 to 1, and t^n to 0 for $n > 0$.

The Kato-Nakayama space X_{\log} is isomorphic to S^1 , and the sheaf $\tau^{-1}\mathcal{A}_X$ is the constant sheaf $\bigoplus_{n \in \mathbb{N}} t^n \cdot \mathbb{C}_{S^1}$. As for \mathcal{A}_{Λ} , we have $\mathcal{A}_{\Lambda} = \bigoplus_{\lambda \in \Lambda} N_{\lambda}$ for $\tau^{-1}\mathbb{C} = \underline{\mathbb{C}}_{S^1}$ -invertible sheaves N_{λ} .

For $\lambda \notin \mathbb{N}$, this line bundle will have a non-trivial monodromy with respect to the action of the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$, and can be described as follows. Consider the universal cover $\pi: \mathbb{R} \rightarrow S^1$, and denote the composite $\mathbb{R} \rightarrow S^1 \rightarrow \text{Spec } \mathbb{C}$ by $\tilde{\tau}$. For every λ , the pullback $\pi^{-1}N_{\lambda}$ is locally constant on \mathbb{R} , hence it is constant, $N_{\lambda} \cong \underline{\mathbb{C}}_{\mathbb{R}}$. Let us formally write t^{λ} for a generator, so that $N_{\lambda} = t^{\lambda} \cdot \underline{\mathbb{C}}_{\mathbb{R}}$. The group \mathbb{Z} of deck transformations of π acts on this sheaf (in the sense that the sheaf has a \mathbb{Z} -equivariant structure), by $k \cdot (t^{\lambda}c) = e^{2\pi i k \lambda} t^{\lambda}c$. With this notation, we can write $\pi^{-1}\mathcal{A}_{\Lambda} = \bigoplus_{\lambda \in \Lambda} t^{\lambda} \cdot \underline{\mathbb{C}}_{\mathbb{R}}$.

Furthermore, recall that $\mathcal{O}_{\Lambda} = \mathcal{A}_{\Lambda} \otimes_{\tau^{-1}\mathcal{A}_X} \tau^{-1}\mathcal{O}_X$. This has the effect (on the universal cover \mathbb{R}) of identifying t^n with its image in $\underline{\mathbb{C}}_{\mathbb{R}} = \tilde{\tau}^{-1}\mathcal{O}_X$, in the description above. Hence, if $\lambda > 1$ and since $\Lambda = \Lambda^{\text{gp}} \cap \mathbb{R}_+$, this forces $t^{\lambda} = 0$, since $t^{\lambda} = t^{\lambda-1} \cdot t = 0$, and t maps to 0 in \mathbb{C} . Hence we have

$$\pi^{-1}\mathcal{O}_{\Lambda} = \bigoplus_{\lambda \in \Lambda \cap [0,1)} t^{\lambda} \cdot \underline{\mathbb{C}}_{\mathbb{R}},$$

where multiplication is determined by $t^\lambda \cdot t^{\lambda'} = t^{\lambda+\lambda'}$ if $\lambda + \lambda' < 1$, and 0 otherwise.

Here t^λ should be thought of as “ z^λ ”, where z is the coordinate of the \mathbb{A}^1 , of which the standard log point is the origin (i.e. the generator of $\overline{M} = \mathbb{N}$).

We will postpone the proof of the following lemma until we discuss local models in Section 4.4 (see Remark 4.13).

Lemma 4.7. *The natural map $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_\Lambda$ induces an isomorphism $\tau^{-1}\mathcal{O}_X^\times \cong \mathcal{O}_\Lambda^\times$.*

One of the main points of this paper is that the ringed space $(X_{\log}, \mathcal{O}_\Lambda)$ can be seen as a sort of root stack of X with respect to coefficients in Λ , for any given quasi-coherent sheaf of monoids $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$. There is a canonical homomorphism of sheaves of monoids $\alpha_\Lambda: M_\Lambda \rightarrow \mathcal{O}_\Lambda$ (see [13, 2.3]) such that $\alpha_\Lambda|_{\mathcal{O}_\Lambda^\times}: \alpha_\Lambda^{-1}\mathcal{O}_\Lambda^\times \rightarrow \mathcal{O}_\Lambda^\times$ is an isomorphism. By Lemma 4.7 we can rewrite extension (4) as

$$0 \longrightarrow \mathcal{O}_\Lambda^\times \longrightarrow M_\Lambda \longrightarrow \tau^{-1}\Lambda \longrightarrow 0$$

and, as described in (4.1), we obtain from this a symmetric monoidal functor $L^\Lambda: \Lambda \rightarrow \text{Div}_{(X_{\log}, \mathcal{O}_\Lambda)}$ from Λ to the stack over opens of X_{\log} of \mathcal{O}_Λ -invertible sheaves with a global section. If N_λ is the invertible sheaf of $\tau^{-1}\mathcal{O}_X$ -modules associated with $\lambda \in \tau^{-1}\Lambda$ via the extension (4) above, then the invertible sheaf of \mathcal{O}_Λ -modules associated with λ via the last extension is canonically isomorphic to $N_\lambda \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_\Lambda$, and hence the corresponding sheaf of \mathcal{O}_Λ -algebras is simply

$$\mathcal{A}_\Lambda \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_\Lambda = \bigoplus_{\tau^{-1}\Lambda} (N_\lambda \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_\Lambda).$$

These data give a sort of log structure on X_{\log} that extends the one of X by adjoining real multiples of the sections of \overline{M} to the structure sheaf. The invertible sheaves $L^\Lambda(\lambda) = (L_\lambda^\Lambda, s_\lambda) \in \text{Div}_{(X_{\log}, \mathcal{O}_\Lambda)}$ for sections $\lambda \in \Lambda$, duals of the sheaves $N_\lambda \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_\Lambda$ mentioned above, will be fundamental for the correspondence with parabolic sheaves. If the log structure is divisorial, morally s_λ should be thought of as f^λ , where f is a local equation of a branch of the boundary divisor, and the sheaf L_λ^Λ is to be thought of as $f^{-\lambda} \cdot \mathcal{O}_\Lambda$.

Example 4.8. If for example $X = \mathbb{A}^1$, then \mathcal{A}_Λ is isomorphic to $\tau^{-1}\mathcal{O}_X$ on opens U that do not contain the origin, and to $\bigoplus_{\lambda \in \Lambda} N_\lambda$ for U containing the origin, and $\mathbb{N} \subseteq \Lambda \subseteq \mathbb{R}_+$. Here the sheaf N_n for $n \in \mathbb{N}$ can be seen as the pullback of $t^n \cdot \mathcal{O}_X$, and in general N_λ can be described, on the universal cover $\mathbb{H} = \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \times S^1 = X_{\log}$ as the sheaf $t^\lambda \cdot \mathcal{O}_X$, with the \mathbb{Z} -equivariant structure determined by $k \cdot t^\lambda = e^{2\pi i k \lambda} t^\lambda$.

The sheaf \mathcal{A}_Λ is the direct sum of these line bundles, and the sheaf $\mathcal{O}_\Lambda = \mathcal{A}_\Lambda \otimes_{\tau^{-1}\mathcal{A}_X} \tau^{-1}\mathcal{O}_X$ can be described on \mathbb{H} by adjoining to \mathcal{O}_X sections t^λ for $\lambda \in \Lambda$, on which \mathbb{Z} acts as above, and with multiplication defined so that $t^\lambda = z^n \cdot t^{\lambda'}$ if $\lambda = n + \lambda'$ and $n \in \mathbb{N}$. Note that the restriction of this description to the origin in \mathbb{A}^1 recovers the discussion of Example 4.6.

For reasons that will be explained later (see Remark 4.20), we need to also tensor the sheaves \mathcal{O}_Λ with the “structure sheaf” \mathcal{O}_X^{\log} of X_{\log} . We briefly recall the construction of this sheaf; see [11, Section 3], [8, Section 1] or [18, Section 3.3] for more details.

The sheaf \mathcal{O}_X^{\log} is the universal sheaf of $\tau^{-1}\mathcal{O}_X$ -algebras with a compatible morphism of sheaves of abelian groups $\mathcal{L} \rightarrow \mathcal{O}_X^{\log}$, where \mathcal{L} is the sheaf of abelian groups of (4.2). An explicit construction is as

$$\mathcal{O}_X^{\log} = (\tau^{-1}\mathcal{O}_X \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}} \mathcal{L}) / \mathcal{I}$$

where \mathcal{I} is the sheaf of ideals generated by local sections of the form $f \otimes 1 - 1 \otimes h(f)$ for $f \in \tau^{-1}\mathcal{O}_X$, and where $h: \tau^{-1}\mathcal{O}_X \rightarrow \mathcal{L}$ is the map in the exact sequence (3), defined by

$$h(f) = (\exp(f), i \operatorname{Im}(f)) \in \mathcal{L} \cong \tau^{-1}M^{\text{gp}} \times_{S_{X_{\log}}^1} i\mathbb{R}_{X_{\log}}.$$

The stalks of \mathcal{O}_X^{\log} can be described as follows: let $y \in X_{\log}$ be a point with image $x = \tau(y)$, and let m_1, \dots, m_r be elements of \mathcal{L}_y , whose image under $\mathcal{L}_y \rightarrow \tau^{-1}\overline{M}_y \cong \overline{M}_x$ is a \mathbb{Z} -basis. Then there is an $\mathcal{O}_{X,x}$ -linear isomorphism $\mathcal{O}_{X,x}[T_1, \dots, T_r] \cong \mathcal{O}_{X,y}^{\log}$ given by $T_i \mapsto m_i$. Hence, morally \mathcal{O}_X^{\log} should be thought of as the sheaf $\tau^{-1}\mathcal{O}_X[\log(m_1), \dots, \log(m_r)]$, where m_1, \dots, m_r are local generators of \overline{M} .

Example 4.9. Assume that X is the standard log point $X = (\operatorname{Spec} \mathbb{C}, \mathbb{N})$. The sheaf \mathcal{O}_X^{\log} on the Kato-Nakayama space S^1 can be described as follows. As usual take the universal cover $\mathbb{R} \rightarrow S^1$, and the constant sheaf of \mathbb{C} -algebras $\underline{\mathbb{C}[T]}_{\mathbb{R}}$, and give it the \mathbb{Z} -equivariant structure determined by $k \cdot T = T - 2k\pi i$. The result of descent to $X_{\log} = S^1$ is the sheaf \mathcal{O}_X^{\log} .

In other words, for every point $y \in S^1$ we have $\mathcal{O}_{X,y}^{\log} \cong \mathbb{C}[T]$, but by moving around the circle once, T becomes $T - 2\pi i$ (this makes sense if we think of T as “ $\log z$ ”, where z is the coordinate of \mathbb{A}^1 , and the standard log point is the origin). For a detailed explanation of the “minus” sign in this action, we refer the reader to [12, Appendix A1].

Note that the global sections of \mathcal{O}_X^{\log} are only the constants, i.e. $\tau_*\mathcal{O}_X^{\log} \cong \mathcal{O}_X$ in this case. This is true also in general (Proposition 4.21).

Remark 4.10. As mentioned in the introduction of [23], it is natural to ask if the map of topological stacks $\phi_{\infty}: X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$ can be promoted in a natural way to a morphism of ringed topological stacks, where we are equipping X_{\log} with the sheaf \mathcal{O}_X^{\log} , and $\sqrt[n]{X}_{\text{top}}$ with its structure sheaf \mathcal{O}_{∞} . The idea here would be the \mathcal{O}_X^{\log} has logarithms of local sections of \overline{M} , so the n -th roots of such sections in the sheaf \mathcal{O}_{∞} should have an image in \mathcal{O}_X^{\log} .

It turns that it is hard to make sense of this if we want a map of rings: if we want to define $\mathcal{O}_{\infty} \rightarrow \mathcal{O}_X^{\log}$ as a sort of logarithm, by sending $t^{\frac{1}{n}}$ to $\frac{1}{n}\log(t)$, then it will not be a homomorphism of rings. A second possibility would be to pass to convergent power series in \mathcal{O}_X^{\log} , and try to impose that $\exp(\frac{1}{n}\log(t)) = \text{image of } t^{\frac{1}{n}}$. This also makes little sense in some situations, because an exponential had better be invertible, but $t^{\frac{1}{n}}$ sometimes is nilpotent, for example if X is the standard log point.

Instead of trying to make sense of this, the solution that we adopt in this paper is to adjoin all needed roots to \mathcal{O}_X^{\log} (as is done in [8, Section 4], for example), in order to have a map as above.

We will set

$$\mathcal{O}_{\Lambda}^{\log} := \mathcal{O}_{\Lambda} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_X^{\log}.$$

This is again a sheaf of rings on X_{\log} , with an injective homomorphism $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{\Lambda}^{\log}$. Morally, on top of adding every possible real power m^{λ} of sections of \overline{M} and exponents in Λ , we are also adding formal logarithms $\log(m)$.

Since the map $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{\Lambda}^{\log}$ induces an isomorphism $\tau^{-1}\mathcal{O}_X^{\times} \cong (\mathcal{O}_{\Lambda}^{\log})^{\times}$ (this is clear from the description of the stalks, and Lemma 4.7), we can lift the symmetric monoidal functor $L^{\Lambda}: \Lambda \rightarrow \operatorname{Div}_{(X_{\log}, \mathcal{O}_{\Lambda})}$ to a symmetric monoidal functor $\Lambda \rightarrow \operatorname{Div}_{(X_{\log}, \mathcal{O}_{\Lambda}^{\log})}$, that we will keep denoting by the same symbol. The line bundle associated with $\lambda \in \Lambda$ via this new symmetric monoidal functor is

simply $L_\lambda^\Lambda \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_X^{\log}$ (and will be denoted again by L_λ^Λ - we will make no use of the invertible sheaf before tensoring with \mathcal{O}_X^{\log}).

The functor $L^\Lambda: \Lambda \rightarrow \text{Div}_{(X_{\log}, \mathcal{O}_\Lambda^{\log})}$ extends the symmetric monoidal functor $L: \overline{M} \rightarrow \text{Div}_X$ on X , so in particular for $\lambda = m \in \overline{M}$ we have $L_m^\Lambda \cong \tau^{-1}L_m \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_\Lambda^{\log}$ (and the sections are also identified). Moreover, as in [5, Proposition 5.2], we get an induced symmetric monoidal functor $\Lambda^{\text{gp}} \rightarrow \text{Pic}_{(X_{\log}, \mathcal{O}_\Lambda^{\log})}$, that we will also denote by L^Λ , by setting $L_{\lambda-\mu}^\Lambda = L_\lambda^\Lambda \otimes_{\mathcal{O}_\Lambda^{\log}} (L_\mu^\Lambda)^\vee$.

Remark 4.11. Some version of the construction of $\mathcal{O}_\Lambda^{\log}$ has appeared in [8]. In (5.1) below we point out that the sheaf of rings $\mathcal{O}_X^{\text{klog}}$ that is used in that paper, and is obtained using the Kummer-étale site of X , is canonically isomorphic to our $\mathcal{O}_{\overline{M}_\mathbb{Q}}^{\log}$. In [18], Ogus uses larger sheaves of rings $\mathcal{O}_X^{\log} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{A}_{\Lambda^{\text{gp}}}$, that are related to our $\mathcal{O}_\Lambda^{\log}$, but not exactly the same.

4.4. Local description. We will need a local description of some of the constructions that we described up to this point.

Let us suppose that X has a Kato chart $X \rightarrow \mathbb{C}(P)$. For the space $\mathbb{C}(P)$ with its natural log structure, we have $\mathbb{C}(P)_{\log} = (\mathbb{R}_{\geq 0} \times S^1)(P) = \text{Hom}(P, \mathbb{R}_{\geq 0} \times S^1)$ and we will also use its universal cover $\widetilde{\mathbb{C}(P)}_{\log} = \mathbb{H}(P) = \text{Hom}(P, \mathbb{H})$ (recall that $\mathbb{H} = \mathbb{R}_{\geq 0} \times \mathbb{R}$). The morphism $\widetilde{\mathbb{C}(P)}_{\log} \rightarrow \mathbb{C}(P)_{\log}$ is induced by the map $\mathbb{H} \rightarrow \mathbb{R}_{\geq 0} \times S^1$ given by $(x, y) \mapsto (x, e^{iy})$, and it is a $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z})$ -principal bundle. Note that $\text{Hom}(P, \mathbb{Z}) = \text{Hom}(P^{\text{gp}}, \mathbb{Z}) \cong \mathbb{Z}^r$ non-canonically, where r is the rank of the free abelian group P^{gp} .

As for the analytic space X , there is a diagram with cartesian squares

$$\begin{array}{ccc} \widetilde{X}_{\log} & \longrightarrow & \mathbb{H}(P) \\ \downarrow & & \downarrow \\ X_{\log} & \longrightarrow & (\mathbb{R}_{\geq 0} \times S^1)(P) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{C}(P) \end{array}$$

where $\widetilde{X}_{\log} \rightarrow X_{\log}$ is a $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z})$ -covering space for the action induced on $\widetilde{X}_{\log} = X \times_{\mathbb{C}(P)} \mathbb{H}(P)$ by the one on the second factor. We will need to consider the analogous constructions of the sheaves \mathcal{A}_Λ and \mathcal{O}_Λ and $\mathcal{O}_\Lambda^{\log}$ on the space \widetilde{X}_{\log} : in this case we will add a tilde to remind ourselves that we are on \widetilde{X}_{\log} rather than on X_{\log} . We will denote by $\pi: \widetilde{X}_{\log} \rightarrow X_{\log}$ the natural map, and by $\tilde{\tau}$ the composite $\tau \circ \pi$.

Note that, since the bottom map of the last diagram is strict, every extension of the form

$$0 \longrightarrow \tau^{-1}\mathcal{O}_X^\times \longrightarrow M_\Lambda \longrightarrow \tau^{-1}\Lambda \longrightarrow 0$$

where Λ is pulled back from a quasi-coherent sheaf of monoids on $\mathbb{C}(P)$, is also pulled back from the analogous extension on $\mathbb{C}(P)_{\log}$. The same is true of the sheaves $\mathcal{A}_\Lambda, \tilde{\mathcal{A}}_\Lambda, \mathcal{O}_\Lambda, \tilde{\mathcal{O}}_\Lambda, \mathcal{O}_\Lambda^{\log}, \tilde{\mathcal{O}}_\Lambda^{\log}$ and of the Deligne–Faltings structure giving the sheaves L_λ^Λ and $\tilde{L}_\lambda^\Lambda$ (both before and after tensoring with \mathcal{O}_X^{\log}).

According to this description of X_{\log} as the quotient of \widetilde{X}_{\log} , we will describe sheaves and maps between sheaves on X_{\log} as objects on \widetilde{X}_{\log} that are $\mathbb{Z}(P)$ -equivariant.

Assume now that $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$ is a quasi-coherent sheaf of monoids, together with a global chart $\Lambda_0 \rightarrow \Lambda(X)$, with $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$. Let us describe the sheaf $\tilde{\mathcal{O}}_{\Lambda}$ in terms of Λ_0 .

Notation 4.12. In order to avoid confusion, in this situation and for $p \in P$ we will denote

- by x^p the element of $\mathbb{C}[P]$, image of p via $P \rightarrow \mathbb{C}[P]$,
- by f_p the section of \mathcal{O}_X , image of p via $P \rightarrow \mathcal{O}_X(X)$,
- by t^p the “placeholder variable” in the sheaf $\mathcal{A}_P = \bigoplus_{p \in P} t^p \cdot \mathcal{O}_X$ on X , and
- by s_p the global section of the invertible sheaf L_P , image of p via $P \rightarrow \text{Div}(X)$.

Consider the sheaf $\tilde{\mathcal{A}}_{\Lambda_0} := \tilde{\mathcal{A}}_P \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]$, where $\mathbb{C}[P] \rightarrow \tilde{\mathcal{A}}_P$ sends x^p to the element t^p of $\tilde{\mathcal{A}}_P = \bigoplus_{p \in P} t^p \cdot \tilde{\tau}^{-1}\mathcal{O}_X$. We also set

$$\tilde{\mathcal{O}}_{\Lambda_0} := \tilde{\mathcal{A}}_{\Lambda_0} \otimes_{\tilde{\mathcal{A}}_P} \tilde{\tau}^{-1}\mathcal{O}_X = \tilde{\tau}^{-1}\mathcal{O}_X \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0],$$

where $\tilde{\mathcal{A}}_P \rightarrow \tilde{\tau}^{-1}\mathcal{O}_X$ sends t^p to the section $f_p \in \tilde{\tau}^{-1}\mathcal{O}_X$, and correspondingly $\mathbb{C}[P] \rightarrow \tilde{\tau}^{-1}\mathcal{O}_X$ sends x^p to f_p . Note that the group $\mathbb{Z}(P)$ acts on these sheaves, by acting on $\mathbb{C}[\Lambda_0]$ via $g \cdot t^{\lambda} = e^{2\pi i g_{\mathbb{R}}(\lambda)} t^{\lambda}$, where here and from now on $g_{\mathbb{R}}: P_{\mathbb{R}} \rightarrow \mathbb{R}$ will denote the natural linear extension of $g: P \rightarrow \mathbb{Z}$. Here t^{λ} should be read as $e^{2\pi i \lambda \cdot \log(t)}$, and the action of g is by “translation” on $\log(t)$.

There are surjective homomorphisms $\tilde{\mathcal{A}}_{\Lambda_0} \rightarrow \tilde{\mathcal{A}}_{\Lambda}$ and $\tilde{\mathcal{O}}_{\Lambda_0} \rightarrow \tilde{\mathcal{O}}_{\Lambda}$, whose kernel is the ideal generated by sections of the form “ $t^{\lambda} - 1$ ” (interpreted in the obvious way in the two sheaves) with λ a local section in the kernel of $(\Lambda_0)_X \rightarrow \Lambda$. Note that this ideal sheaf is not quasi-coherent, because the rings $\tilde{\mathcal{A}}_{\Lambda}$ and $\tilde{\mathcal{O}}_{\Lambda}$ are not quasi-coherent (see Example 4.1). This gives an explicit description for the sheaves $\tilde{\mathcal{A}}_{\Lambda}$ and $\tilde{\mathcal{O}}_{\Lambda}$, similar to the one we obtained above for $\tilde{\mathcal{A}}_{\Lambda_0}$ and $\tilde{\mathcal{O}}_{\Lambda_0}$, where the monoid Λ_0 is replaced by the sheaf Λ .

Remark 4.13. If $X = \mathbb{C}(P)$ and Λ has a chart $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$, we can describe \mathcal{O}_{Λ} on X_{\log} more concretely, in the style of [8, (1.1) and (3.2)].

Let $U = (\text{Spec } \mathbb{C}[P^{\text{gp}}])_{\text{an}} \subseteq X$, and note that this embedding lifts to $j: U \subseteq X_{\log}$. Moreover the constant sheaf $\underline{P^{\text{gp}}}_{X_{\log}}$ can be seen as a subsheaf of $j_*\mathcal{O}_U^{\times}$. Then the sheaf \mathcal{O}_{Λ} can be identified with the subsheaf of rings of $j_*\mathcal{O}_U$, generated by $\tau^{-1}\mathcal{O}_X$ and by local sections of the form p^{α} , for $p \otimes \alpha \in \Lambda \subseteq \overline{M} \otimes \mathbb{R}_+$, and where p is seen as a section of $\underline{P^{\text{gp}}}_{X_{\log}} \subseteq j_*\mathcal{O}_U^{\times}$.

In the same manner, the sheaf $\mathcal{O}_{\Lambda}^{\log}$ is obtained by adding, on top of the previous sections, also local logarithms $\log(p)$ for sections of $\underline{P^{\text{gp}}}_{X_{\log}}$. For $\Lambda = \overline{M}_{\mathbb{Q}}$, this coincides with the description of $\mathcal{O}_X^{\text{klog}}$ in [8, 3.2].

From this description, it is clear that the inclusion $\tau^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{\Lambda}$ induces an isomorphism $\tau^{-1}\mathcal{O}_X^{\times} \cong \mathcal{O}_{\Lambda}^{\times}$ between the units: if a local section of the form p^{α} were invertible, then also p would be. But if p does not become zero in \overline{M} (locally around a point), then it cannot be invertible there. Moreover, this assertion for $X = \mathbb{C}(P)$ implies the same for any fine saturated log scheme (stated in Lemma 4.7).

We can also describe the invertible sheaf $\tilde{L}_{\Lambda}^{\Lambda}$ on \tilde{X}_{\log} (before tensoring with \mathcal{O}_X^{\log}): this is an invertible sheaf of $\tilde{\mathcal{O}}_{\Lambda}$ -modules, which is trivial, and we will think about as $\tilde{L}_{\Lambda}^{\Lambda} = t^{-\lambda} \cdot \tilde{\mathcal{O}}_{\Lambda}$. This sheaf also has a natural $\mathbb{Z}(P)$ -equivariant structure, a “shifted” version of the one of $\tilde{\mathcal{O}}_{\Lambda}$, and the sheaf on X_{\log} obtained by descent is L_{Λ}^{Λ} . The global section \tilde{s}_{λ} , given by the natural map $\tilde{\mathcal{O}}_{\Lambda} \rightarrow \tilde{L}_{\Lambda}^{\Lambda}$ (that can be seen as multiplication by $t^{\lambda} \in \tilde{\mathcal{O}}_{\Lambda}$), also descends to the global section s_{λ} of L_{Λ}^{Λ} . Note that as sheaves of $\tilde{\mathcal{O}}_{\Lambda}$ -modules, we have $\tilde{L}_{\Lambda}^{\Lambda} \cong \tilde{\mathcal{O}}_{\Lambda}$, but the action of $\mathbb{Z}(P)$ is different, unless $\bar{\lambda} = 0$ in $\Lambda^{\text{gp}}/\overline{M}^{\text{gp}}$, i.e. $\lambda \in \overline{M}^{\text{gp}}$.

Observe also that if λ is a section m of \overline{M}^{gp} , then $\tilde{L}_\lambda^\Lambda = t^{-m} \cdot \tilde{\mathcal{O}}_\Lambda \cong \tilde{\tau}^{-1} L_m \otimes_{\tilde{\tau}^{-1} \mathcal{O}_X} \tilde{\mathcal{O}}_\Lambda$, where $L: \overline{M}^{\text{gp}} \rightarrow \text{Pic}_X$ is the functor associated with the Deligne–Faltings structure of the log analytic space X .

Now we bring the sheaf \mathcal{O}_X^{\log} into the picture. On \tilde{X}_{\log} we can tensor the sheaf $\tilde{\mathcal{O}}_\Lambda$ and the various line bundles $\tilde{L}_\lambda^\Lambda$ with $\tilde{\mathcal{O}}_X^{\log}$ over $\tilde{\tau}^{-1} \mathcal{O}_X$ to obtain $\tilde{\mathcal{O}}_\Lambda^{\log}$ and the $\tilde{\mathcal{O}}_\Lambda^{\log}$ -line bundles $\tilde{L}_\lambda^\Lambda \otimes_{\tilde{\tau}^{-1} \mathcal{O}_X} \tilde{\mathcal{O}}_X^{\log}$ (along with the induced global sections), that, as in the previous section, we will continue to denote by $\tilde{L}_\lambda^\Lambda$.

The local descriptions of these sheaves are obtained from the ones described above, by tensoring with the sheaf $\tilde{\mathcal{O}}_X^{\log}$ on \tilde{X}_{\log} . A description of this latter sheaf is given in [18, Lemma 3.3.4]: if \mathcal{I} denotes the sheaf of ideals in $\tilde{\tau}^{-1} \mathcal{O}_X \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}} P^{\text{gp}}$ generated by elements of the form $f_p \otimes 1 - 1 \otimes p$ for p a local section of P that maps to a unit in M (via the chart morphism $\underline{P} \rightarrow M$), then we have an isomorphism

$$\tilde{\mathcal{O}}_X^{\log} \cong (\tilde{\tau}^{-1} \mathcal{O}_X \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}} P^{\text{gp}}) / \mathcal{I}.$$

The resulting sheaves have an induced $\mathbb{Z}(P)$ -action, and the results of descent to X_{\log} are the sheaf $\mathcal{O}_\Lambda^{\log}$ and the line bundles L_λ^Λ .

Example 4.14. The description of the sheaf $\tilde{\mathcal{O}}_X^{\log}$ can be made more explicit on the fibers of $\tilde{\tau}: \tilde{X}_{\log} \rightarrow X$ (generalizing Example 4.9). Fix $x \in X$, and call $P = \overline{M}_x$. Then we can find a Kato chart with monoid P for X around x .

Taking the fibers over x , we can write $(X_{\log})_x \cong \text{Hom}(P^{\text{gp}}, S^1)$ and $(\tilde{X}_{\log})_x \cong \text{Hom}(P^{\text{gp}}, \mathbb{R})$. If we fix an isomorphism $P^{\text{gp}} \cong \mathbb{Z}^r$, we are looking at the universal cover $\mathbb{R}^r \rightarrow (S^1)^r$. The sheaf $\tilde{\mathcal{O}}_X^{\log}$ on \mathbb{R}^r is the constant sheaf $\mathbb{C}[T_1, \dots, T_r]_{\mathbb{R}^r}$, and it has a \mathbb{Z}^r -equivariant structure, where the i -th standard generator e_i acts on T_j by sending it to $T_j - 2\pi i \delta_{ij}$, where δ_{ij} is the usual Kronecker delta. By descending this to $(S^1)^r$, we obtain the sheaf \mathcal{O}_X^{\log} .

4.5. Sheaves of modules on X_{\log} . The last ingredient that we need in order to discuss the correspondence with parabolic sheaves is a discussion of quasi-coherent sheaves of modules on X_{\log} and their properties.

In general if (T, \mathcal{O}_T) is a ringed space, the usual meaning for “quasi-coherent sheaf of \mathcal{O}_T -modules” refers to the existence of local presentations

$$\mathcal{O}_T^{\oplus J} \longrightarrow \mathcal{O}_T^{\oplus I} \longrightarrow F \longrightarrow 0$$

with possibly infinite index sets I, J . In complete generality, it is not clear how well-behaved the category of such sheaves is.

We will instead adopt the terminology of [7, Section 2.1] (as for parabolic sheaves, see Remark 3.19). For quasi-coherence of sheaves of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} , we will use the following definition. As usual we denote by $\tau: X_{\log} \rightarrow X$ the projection.

Definition 4.15. We will say that a sheaf F of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} is *finitely presented* if locally on X (i.e. locally on X_{\log} for open sets of the form $\tau^{-1}U$, with $U \subseteq X$ open) it admits a presentation of the form

$$\bigoplus_j L_{\lambda_j}^\Lambda \longrightarrow \bigoplus_i L_{\lambda_i}^\Lambda \longrightarrow F \longrightarrow 0$$

for which the index sets for i and j are finite.

We will say that a sheaf F of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} is *quasi-coherent* if locally on X it can be written as filtered colimit of finitely presented sheaves.

Here the sheaves L_λ^Λ are the line bundles on X_{\log} of (4.3).

Remark 4.16. We refrain from using the term “coherent” for the sheaves that locally admit finite presentations, because already on the infinite root stack, the structure sheaf might not be coherent (see [22, Example 4.17]), so that “finitely presented” and “coherent” are not equivalent notions. We expect the same to happen in this context. Note that, since \mathcal{O}_X is coherent, on X itself it is indeed true that finite presentation and coherence are equivalent.

We will denote the category of finitely presented sheaves of $\mathcal{O}_\Lambda^{\log}$ -modules by $\mathrm{FP}_\Lambda(X)$, and the category of quasi-coherent sheaves by $\mathrm{Qcoh}_\Lambda(X)$.

Remark 4.17. Some comments about this definition are in order.

First of all, note that the line bundles L_λ^Λ are locally isomorphic to $\mathcal{O}_\Lambda^{\log}$ on X_{\log} , so a finitely presented sheaf as we defined it will also be finitely presented in the “standard” sense (of admitting local presentations as a cokernel of a map between free sheaves) on the ringed space $(X_{\log}, \mathcal{O}_\Lambda^{\log})$. In view of the correspondence with parabolic sheaves, though, we want to restrict to the class that admit local presentations *on opens pulled back from X* . Once we choose to do this, using direct sums of the non-trivial line bundles L_λ^Λ is forced.

This condition on having presentations for a topology of X_{\log} that is coarser than the natural one should be compared with the situation of root stacks: the map $\sqrt[n]{X} \rightarrow X$ is a homeomorphism on the associated topological spaces, and even though one can localize in the étale topology around points of $\sqrt[n]{X}$ where there are non-trivial stabilizers, one can not “physically” localize on the fibers (since they are single points!), as one can do on the Kato-Nakayama space.

We will talk about exact sequences of sheaves in $\mathrm{Qcoh}_\Lambda(X)$ or $\mathrm{FP}_\Lambda(X)$, meaning that the same sequence is exact when viewed in $\mathrm{Mod}(\mathcal{O}_\Lambda^{\log})$.

Let us consider pullback and pushforward along the morphism of ringed spaces $\tau: (X_{\log}, \mathcal{O}_\Lambda^{\log}) \rightarrow X$. We will omit the sheaf of weights Λ from the notation of those functors, since there will be no risk of confusion.

We can define pullback $\tau^*: \mathrm{Mod}(\mathcal{O}_X) \rightarrow \mathrm{Mod}(\mathcal{O}_\Lambda^{\log})$ and pushforward $\tau_*: \mathrm{Mod}(\mathcal{O}_\Lambda^{\log}) \rightarrow \mathrm{Mod}(\mathcal{O}_X)$ as usual. Since $\tau^*\mathcal{O}_X \cong \mathcal{O}_\Lambda^{\log}$ and τ^* commutes with colimits and is right exact, it is also clear that the pullback functor will restrict nicely to the subcategories of quasi-coherent and finitely presented sheaves, inducing functors $\tau^*: \mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}_\Lambda(X)$ and $\tau^*: \mathrm{FP}(X) \rightarrow \mathrm{FP}_\Lambda(X)$. It is less clear that the pushforward will behave well. This is what the rest of this section will be about.

We will start by showing that the functor $\tau_*: \mathrm{Qcoh}_\Lambda(X) \rightarrow \mathrm{Mod}(\mathcal{O}_X)$ is exact. Note that by standard arguments we can define a derived pushforward functor $R\tau_*: D^+(\mathrm{Mod}(\mathcal{O}_\Lambda^{\log})) \rightarrow D^+(\mathrm{Mod}(\mathcal{O}_X))$.

Lemma 4.18. *Let X be a logarithmic point $(\mathrm{Spec} \mathbb{C}, P)$, with P a toric monoid. Then for every $\lambda \in \Lambda^{\mathrm{gp}}$ and $m > 0$, we have $H^m(X_{\log}, L_\lambda^\Lambda) = 0$.*

Proof. This was proven in [8, Proposition 3.7], [15, Proposition 4.6] and [12, Proposition 2.2.10] in the case where X is a general fine saturated log analytic space, $\Lambda = \overline{M}$ and $\lambda = 0$, so that there are no “added roots”, and $L_\lambda^\Lambda = \mathcal{O}_X^{\log}$.

The proof in this case is along the same lines of the one of [12, Proposition 2.2.10], so we just sketch it briefly. Call $p_i \in P$ elements that give a \mathbb{Z} -basis of $\overline{M}_x^{\mathrm{gp}} = P^{\mathrm{gp}}$. We have an isomorphism

$$L_\lambda^\Lambda \cong t^{-\lambda} \cdot \mathbb{C}[\log(p_i), p_i^{\lambda_{ij}}]$$

where both $\log(p_i)$ and $p_i^{\lambda_{i_j}}$ denote local sections. Here the $p_i^{\lambda_{i_j}}$ appearing are the ones for which $p_i \otimes \lambda_{i_j} \in \Lambda \subseteq P_{\mathbb{R}}$. Since this sheaf is locally constant on X_{\log} , we have

$$H^m(X_{\log}, L_{\lambda}^{\Lambda}) = H^m(\mathbb{Z}(P), (L_{\lambda}^{\Lambda})_y)$$

where y is a fixed point of X_{\log} , and the right term is group cohomology, with respect to the action of the fundamental group $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z})$ of X_{\log} .

Now since $(L_{\lambda}^{\Lambda})_y$, as a \mathbb{C} -vector space, is a direct sum of copies of

$$t^{-\lambda} \cdot p_i^{\lambda_{i_j}} \cdot \mathbb{C}[\log(p_1), \dots, \log(p_r)] \cong \mathbb{C}[\log(p_1), \dots, \log(p_r)]$$

where the action of the generator $\gamma_i \in \mathbb{Z}(P)$ dual to the element q_i is given by

$$\gamma_i(\log(p_j)) = e^{2\pi i(\gamma_i)_{\mathbb{R}}(\lambda_{i_j} - \lambda)}(\log(p_j) - 2\pi i\delta_{ij}),$$

the conclusion follows exactly from the same proof as in [12, Proposition 2.2.10]. \square

Proposition 4.19. *The functor $\tau_*: \text{Qcoh}_{\Lambda}(X) \rightarrow \text{Mod}(\mathcal{O}_X)$ is exact.*

Proof. As usual the pushforward is left exact. We will show that for every quasi-coherent sheaf $F \in \text{Qcoh}_{\Lambda}(X)$ we have $R^1\tau_*F = 0$ (where the derived functor is computed in $\text{Mod}(\mathcal{O}_{\Lambda}^{\log})$), and this will imply the exactness. Since $R^1\tau_*$ commutes with filtered colimits, we can assume that F is finitely presented.

In order to show $R^1\tau_*F = 0$, let us fix a point $x \in X$ and check that the stalk $(R^1\tau_*F)_x$ is zero. By proper base change (as formulated for example in [12, Appendix A2]) via the cartesian diagram

$$\begin{array}{ccc} (X_{\log})_x \cong (x)_{\log} & \longrightarrow & X_{\log} \\ \downarrow & & \downarrow \\ x & \longrightarrow & X \end{array}$$

and Nakayama's lemma, we can assume that X is a logarithmic point $X = x = (\text{Spec } \mathbb{C}, P)$.

We have to check that if F is a quasi-coherent sheaf of $\mathcal{O}_{\Lambda}^{\log}$ -modules on X_{\log} , then $R^1\tau_*F = H^1(X_{\log}, F) = 0$. Since F is finitely presented, we have a presentation

$$\bigoplus_j L_{\lambda_j}^{\Lambda} \longrightarrow \bigoplus_i L_{\lambda_i}^{\Lambda} \xrightarrow{f} F \longrightarrow 0,$$

with finitely many summands on the whole X_{\log} , and by Lemma 4.18 we have $H^i(X_{\log}, L_{\lambda}^{\Lambda}) = 0$ for every λ and $i > 0$.

Let us consider the kernel K_1 of the map f in the presentation above, fitting in a short exact sequence

$$0 \longrightarrow K_1 \longrightarrow \bigoplus_i L_{\lambda_i}^{\Lambda} \xrightarrow{f} F \longrightarrow 0,$$

and the induced the long exact sequence in cohomology. Using the fact that $H^i(X_{\log}, L_{\lambda}^{\Lambda}) = 0$ for $i > 0$ and any λ , we see that $H^i(X_{\log}, F) \cong H^{i+1}(X_{\log}, K_1)$ for $i > 0$. We claim that K_1 also has a presentation of the form

$$\bigoplus_k L_{\lambda_k}^{\Lambda} \longrightarrow \bigoplus_j L_{\lambda_j}^{\Lambda} \longrightarrow K_1 \longrightarrow 0,$$

where $\bigoplus_k L_{\lambda_k}^{\Lambda}$ might have infinitely many summands.

Let us write $\sum_i a_{i,j} t^{-\lambda_i}$ for the image of $t^{-\lambda_j}$ via $\bigoplus_j L_{\lambda_j}^\Lambda \rightarrow \bigoplus_i L_{\lambda_i}^\Lambda$. Assume that a local section $s = \sum_j b_j t^{-\lambda_j}$ is in the kernel of that map, i.e. we have

$$\sum_j b_j \sum_i a_{i,j} t^{-\lambda_i} = \sum_i \left(\sum_j b_j a_{i,j} \right) t^{-\lambda_i} = 0.$$

Here $a_{i,j}$ and b_j are complex numbers.

We want to argue that s is in the image of a map $\bigoplus_\alpha L_{\lambda_\alpha}^\Lambda \rightarrow \bigoplus_j L_{\lambda_j}^\Lambda$ that completely lands in the kernel, by using induction on the number of non-zero summands in s . Fix λ_{j_1} such that $b_{j_1} \neq 0$. If $t^{-\lambda_{j_1}}$ is mapped to the zero section, we are done, otherwise there exists i_1 with $a_{i_1,j_1} \neq 0$. Looking at the coordinate i_1 in the target, we see that $\sum_j b_j a_{i_1,j} = 0$, and since $a_{i_1,j_1} \neq 0$ we can solve for b_{j_1} . This gives us a section s' in the kernel of $\bigoplus_j L_{\lambda_j}^\Lambda \rightarrow \bigoplus_i L_{\lambda_i}^\Lambda$ and in the image of a map $L_{\lambda_{j_1}}^\Lambda \rightarrow \bigoplus_j L_{\lambda_j}^\Lambda$ landing in the kernel, such that $s - s'$ has strictly fewer non-zero terms than s . By induction, s will be in the image of a map of the form $\bigoplus_\alpha L_{\lambda_\alpha}^\Lambda \rightarrow \bigoplus_j L_{\lambda_j}^\Lambda$. By taking a big direct sum over sections in the kernel of $\bigoplus_j L_{\lambda_j}^\Lambda \rightarrow \bigoplus_i L_{\lambda_i}^\Lambda$, we obtain the desired presentation.

Back to sheaf cohomology, we can now iterate the argument above and obtain a chain of isomorphisms $H^i(X_{\log}, F) \cong H^{i+1}(X_{\log}, K_1) \cong H^{i+2}(X_{\log}, K_2) \cong \dots \cong H^{i+k}(X_{\log}, K_k)$ for $i > 0$. As soon as $1 + k > n$, the last cohomology group has to vanish, and hence we get $H^1(X_{\log}, F) = 0$, as we wanted to prove. \square

Remark 4.20. The previous proposition is the reason why it is important to bring the sheaf \mathcal{O}_X^{\log} into the picture: without tensoring the other sheaves by it, this proposition does not hold.

For example, consider the standard log point $(\text{Spec } \mathbb{C}, \mathbb{N})$, with Kato-Nakayama space $\tau: S^1 \rightarrow \text{Spec } \mathbb{C}$. The structure sheaf downstairs is \mathbb{C} , and its pullback is the constant sheaf $\underline{\mathbb{C}}_{S^1}$. Clearly

$$R^1 \tau_* \tau^{-1} \mathcal{O}_{\text{Spec } \mathbb{C}} = R^1 \tau_* \underline{\mathbb{C}}_{S^1} = H^1(S^1, \mathbb{C}) \neq 0$$

in this case. On the other hand we do have $R^1 \tau_*(\tau^{-1} \mathbb{C} \otimes_{\tau^{-1} \mathbb{C}} \mathcal{O}_{\text{Spec } \mathbb{C}}^{\log}) = 0$.

The heuristic here is that the non-trivial geometry that is introduced by the Kato-Nakayama construction obstructs the exactness of τ_* , and tensoring with the sheaf \mathcal{O}_X^{\log} (which has sections that interact with this geometry) balances this out.

Without exactness, it might still be that part of the arguments go through, but for example it is not clear that the equivalence between parabolic sheaves and sheaves on X_{\log} would respect exactness, something which is certainly desirable.

Now we can deduce that τ_* respects quasi-coherence and finite presentation.

Lemma 4.21. *The natural morphism $\mathcal{O}_X \rightarrow \tau_* \mathcal{O}_\Lambda^{\log}$ is an isomorphism, and for every $\lambda \in \Lambda(X)$ the sheaf $\tau_* L_\lambda^\Lambda$ is finitely presented.*

Proof. The first assertion was proven in [8] in the case $\Lambda = \overline{M}$. In the general case the proof is similar.

For the second point, we can localize where X and Λ have charts $P \rightarrow \overline{M}(X)$ and $\Lambda_0 \rightarrow \Lambda(X)$ with $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$. We claim that for every $\lambda \in \Lambda_0$ there is an isomorphism

$$\tau_* L_\lambda^\Lambda \cong \varinjlim_{P \ni p \leq \lambda} L_p$$

where the map $L_p \rightarrow L_{p'}$ for $p \leq p'$, i.e. $p' = p + p''$ for some $p'' \in P$ and $L_{p'} \cong L_p \otimes_{\mathcal{O}_X} L_{p''}$, is given by multiplication by the section $s_{p''} \in \Gamma(L_{p''})$. After we prove this, it is sufficient to note that this is a finite colimit (because P is finitely generated) of coherent sheaves, hence coherent itself.

To prove the claim, note that there is a natural map $\tau^*L_p \rightarrow L_\lambda^\Lambda$ for every $p \leq \lambda$, that by the projection formula gives $L_p \rightarrow \tau_*L_\lambda^\Lambda$. Taking the colimit we obtain a map $\varinjlim_{P \ni p \leq \lambda} L_p \rightarrow \tau_*L_\lambda^\Lambda$. Now note that this map can be seen as the natural map

$$\varinjlim_{P \ni p \leq \lambda} t^{-p}\mathcal{O}_X \rightarrow (t^{-\lambda}\tilde{\mathcal{O}}_\Lambda^{\log})_0$$

where $(-)_0$ denotes taking the global sections that are invariant with respect to the the action of the group $\mathbb{Z}(P)$. It is clear that this map is an isomorphism. \square

Proposition 4.22. *The functor $\tau_*: \text{Mod}(\mathcal{O}_\Lambda^{\log}) \rightarrow \text{Mod}(\mathcal{O}_X)$ sends $\text{Qcoh}_\Lambda(X)$ into $\text{Qcoh}(X)$ and $\text{FP}_\Lambda(X)$ into $\text{FP}(X)$.*

Proof. First note that, since τ is a proper map of topological spaces, the functor τ_* commutes with filtered colimits.

Since $\tau_*: \text{Qcoh}_\Lambda(X) \rightarrow \text{Mod}(\mathcal{O}_X)$ is exact (Proposition 4.19) and $\tau_*L_\lambda^\Lambda$ is finitely presented (Lemma 4.21), a local presentation for $F \in \text{FP}_\Lambda(X)$ gives a local presentation of τ_*F as a cokernel of a map of coherent sheaves of \mathcal{O}_X -modules, so $\tau_*F \in \text{Coh}(X)$. The fact that τ_* commutes with filtered colimits lets us conclude also that $\tau_*\text{Qcoh}_\Lambda(X) \subseteq \text{Qcoh}(X)$. \square

To conclude this section, we point out that the projection formula holds for the map $\tau: X_{\log} \rightarrow X$ and quasi-coherent sheaves on Kato-Nakayama space. This is in analogy with the projection formula for the root stacks of [21, Proposition 2.2.10] and [22, Proposition 4.16].

Proposition 4.23 (Projection formula). *Let X be a fine saturated log analytic space and $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$ be a quasi-coherent sheaf of monoids. Then:*

- For $F \in \text{Qcoh}_\Lambda(X)$ and $G \in \text{Qcoh}(X)$, the natural map

$$\tau_*F \otimes_{\mathcal{O}_X} G \rightarrow \tau_*(F \otimes_{\mathcal{O}_\Lambda^{\log}} \tau^*G)$$

is an isomorphism, and

- for $G \in \text{Qcoh}(X)$ the natural map $G \rightarrow \tau_*\tau^*G$ is an isomorphism.

The last item has already been proven, in the case where $\Lambda = \overline{M}$, in [8, Proposition 3.7 (3)].

Proof. This follows formally from exactness of τ_* (Proposition 4.19) and the fact that $\tau_*\mathcal{O}_\Lambda^{\log} \cong \mathcal{O}_X$ (Lemma 4.21), for example as in [19, Corollary 5.3]. \square

5. THE CORRESPONDENCE

We are now ready to state and prove the main result of this paper.

Theorem 5.1. *Let X be a fine saturated log analytic space, with log structure $\alpha: M \rightarrow \mathcal{O}_X$. Then for every quasi-coherent sheaf of monoids $\overline{M} \subseteq \Lambda \subseteq \overline{M}_{\mathbb{R}}$ there is an equivalence of categories $\Phi: \text{Qcoh}_\Lambda(X) \rightarrow \text{Par}(X, \Lambda)$. Moreover, the equivalence respects exactness, and restricts to the subcategories of finitely presented sheaves $\text{FP}_\Lambda(X)$ and $\text{FPPar}(X, \Lambda)$.*

The proof will be an adaptation of the one of [5, Theorem 6.1], that we briefly recalled in (3.4) above.

Proof. We will proceed in a few steps.

Construction of Φ :

Let us construct the functor Φ . Recall that on X_{\log} we have a symmetric monoidal functor $L^\Lambda: \Lambda^{\text{gp}} \rightarrow \text{Div}_{(X_{\log}, \mathcal{O}_\Lambda^{\log})}$ (see (4.3)). Assume that we are given a quasi-coherent sheaf $F \in$

$\mathrm{Qcoh}_\Lambda(X)$ on X_{\log} , and we want to produce a parabolic sheaf with weights in Λ . Suppose $U \subseteq X$ is open, and take an object $a \in \Lambda^{\mathrm{wt}}(U)$. Set

$$\Phi(F)_a := \tau_*(F \otimes_{\mathcal{O}_\Lambda^{\log}} L_a^\Lambda).$$

For an arrow $a \rightarrow b$ in $\Lambda^{\mathrm{wt}}(U)$ corresponding to $\lambda \in \Lambda(U)$, we define $\Phi(F)_a \rightarrow \Phi(F)_b$ to be the morphism induced by multiplication by the section s_λ from $L_a^\Lambda \cong L_a^\Lambda \otimes_{\mathcal{O}_\Lambda^{\log}} L_0^\Lambda \rightarrow L_a^\Lambda \otimes_{\mathcal{O}_\Lambda^{\log}} L_\lambda^\Lambda \cong L_b^\Lambda$ by tensoring by F and pushing forward to X .

If $a = b + m$ with $m \in \overline{M}^{\mathrm{gp}}(U)$, we have

$$\begin{aligned} \Phi(F)_a &= \Phi(F)_{b+m} = \tau_*(F \otimes_{\mathcal{O}_\Lambda^{\log}} L_{b+m}^\Lambda) \\ &\cong \tau_*(F \otimes_{\mathcal{O}_\Lambda^{\log}} L_b^\Lambda \otimes_{\mathcal{O}_\Lambda^{\log}} \tau^* L_m) \\ &\cong \tau_*(F \otimes_{\mathcal{O}_\Lambda^{\log}} L_b^\Lambda) \otimes_{\mathcal{O}_X} L_m \\ &= \Phi(F)_b \otimes_{\mathcal{O}_X} L_m, \end{aligned}$$

where we used the projection formula for τ . This gives the required isomorphism $\rho_{m,a}^{\Phi(F)}$, and the map $\Phi(F)_b \rightarrow \Phi(F)_a$ corresponds to multiplication by the section s_m of L_m .

If $V \subseteq U \subseteq X$, then it is clear that $\Phi(F)_a|_V \cong \Phi(F|_{V_{\log}})_{a|_V}$ canonically, and this restriction is also compatible with the isomorphisms $\rho_{m,a}^{\Phi(F)}$. The other conditions in the definition of a parabolic sheaf are easily verified.

Finally, we check that $\Phi(F)$ is a quasi-coherent parabolic sheaf. This is a local question on X , so we can assume that F is a filtered colimit of finitely presented sheaves, as in Definition 4.15. Assume for the time being that we have proven that Φ sends finitely presented sheaves to finitely presented parabolic sheaves (Definition 3.15). Then $\Phi(F)$ will be a filtered colimit of finitely presented parabolic sheaves, and hence quasi-coherent. We will verify the assertion about finitely presented sheaves at the end of the proof.

We leave the construction of the action of the functor $\Phi: \mathrm{Qcoh}_\Lambda(X) \rightarrow \mathrm{Par}(X, \Lambda)$ on arrows to the reader.

Local construction of the quasi-inverse Ψ :

To prove that the functor Φ is an equivalence we will construct an inverse locally on X (observe that both $\mathrm{Qcoh}_{\Lambda|_-}(-)$ and $\mathrm{Par}(-, \Lambda|_-)$ are stacks on the classical site of X). So we may assume that X has a Kato chart $X \rightarrow \mathbb{C}(P)$ for a toric monoid P , and that Λ has a compatible chart $\Lambda_0 \rightarrow \Lambda(X)$, with $P \subseteq \Lambda_0 \subseteq P_{\mathbb{R}}$.

In this case we have fairly explicit descriptions of the sheaves \mathcal{A}_Λ and \mathcal{O}_Λ in terms of their pullback to the covering space \tilde{X}_{\log} of X_{\log} . Recall that in this situation we have a cartesian diagram

$$\begin{array}{ccc} \tilde{X}_{\log} & \longrightarrow & \mathbb{H}(P) \\ \downarrow & & \downarrow \\ X_{\log} & \longrightarrow & (\mathbb{R}_{\geq 0} \times S^1)(P) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{C}(P) \end{array}$$

where the two top vertical maps are covering spaces for the group $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z})$. Recall also from Proposition 3.20 that there is a natural equivalence $\text{Par}(X, \Lambda) \cong \text{Par}(X, \Lambda_0)$

Let us assume that $E: \Lambda_0^{\text{wt}} \rightarrow \text{Qcoh}(X)$ is a parabolic sheaf for Λ_0 . We will produce a sheaf of $\tilde{\mathcal{O}}_\Lambda^{\log}$ -modules on \tilde{X}_{\log} equipped with a $\mathbb{Z}(P)$ -equivariant structure. This will give a sheaf of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} by descent.

Recall that $\tilde{\tau}: \tilde{X}_{\log} \rightarrow X$ denotes the natural projection. Also, in this situation the sheaf $\tilde{\mathcal{O}}_\Lambda$ is a quotient of the sheaf

$$\tilde{\mathcal{O}}_{\Lambda_0} = \tilde{\tau}^{-1} \mathcal{O}_X \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0],$$

where $\mathbb{C}[P] \rightarrow \tilde{\tau}^{-1} \mathcal{O}_X$ is obtained from the map $X \rightarrow \mathbb{C}(P)$. The kernel of $\tilde{\mathcal{O}}_{\Lambda_0} \rightarrow \tilde{\mathcal{O}}_\Lambda$ is locally generated by elements of the form $t^\lambda - 1$, where λ is in the kernel of the map of sheaves of monoids $(\Lambda_0)_X \rightarrow \Lambda$.

Starting from E , we consider the direct sum $\bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} E_\lambda$ as a sheaf of \mathcal{O}_X -modules on X . We pull this back to \tilde{X}_{\log} and obtain

$$\tilde{E} := \bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} \tilde{\tau}^{-1} E_\lambda,$$

which is a sheaf of $\tilde{\tau}^{-1} \mathcal{O}_X$ -modules.

Consider the sheaf of $\tilde{\tau}^{-1} \mathcal{O}_X$ -algebras $A := \bigoplus_{a \in P^{\text{gp}}} \tilde{\tau}^{-1} L_a \cong \bigoplus_{a \in P^{\text{gp}}} t^{-a} \cdot \tilde{\tau}^{-1} \mathcal{O}_X$, where t^{-a} is just a placeholder variable. First note that, on top of its natural $\tilde{\tau}^{-1} \mathcal{O}_X$ -module structure, \tilde{E} is also a sheaf of A -modules, via the map $\tilde{E} \otimes_{\mathcal{O}_X} A \rightarrow \tilde{E}$ constructed as follows: we can define

$$\tilde{E} \otimes_{\tilde{\tau}^{-1} \mathcal{O}_X} A = \bigoplus_{a \in P^{\text{gp}}, \lambda \in \Lambda_0^{\text{gp}}} (\tilde{\tau}^{-1} E_\lambda \otimes_{\tilde{\tau}^{-1} \mathcal{O}_X} \tilde{\tau}^{-1} L_a) \rightarrow \tilde{E} = \bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} \tilde{\tau}^{-1} E_\lambda$$

by using the pullback via $\tilde{\tau}$ of the given isomorphisms $E_\lambda \otimes_{\mathcal{O}_X} L_a \cong E_{\lambda+a}$ for $a \in P^{\text{gp}}$ and $\lambda \in \Lambda_0^{\text{gp}}$.

The sheaf \tilde{E} on \tilde{X}_{\log} has a $\mathbb{Z}(P)$ -equivariant structure: if $g \in \mathbb{Z}(P)$ and f is a section of $\tilde{\tau}^{-1} E_q$, we define $g \cdot f = e^{2\pi i g_{\mathbb{R}}(q)} f$ as a section of $\tilde{\tau}^{-1} E_q$. Moreover, \tilde{E} has an action of the sheaf $\mathbb{C}[\Lambda_0]$ on \tilde{X}_{\log} , compatible (by property (a) in Definition 3.5) with the one of $\mathbb{C}[P]$ induced by $\mathbb{C}[P] \rightarrow A$, where this map sends x^p to the section $t^{-p} \cdot f_p \in t^{-p} \cdot \tilde{\tau}^{-1} \mathcal{O}_X \subseteq A$: for a section x^λ of $\mathbb{C}[\Lambda_0]$, we define the action on the piece $\tilde{\tau}^{-1} E_\mu$ to be the pullback via $\tilde{\tau}$ of the map $E_\mu \rightarrow E_{\mu+\lambda}$ coming from the structure of parabolic sheaf. Hence \tilde{E} is a $\mathbb{Z}(P)$ -equivariant sheaf of $A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]$ -modules.

Now consider the morphism of sheaves of algebras $A \rightarrow \tilde{\mathcal{O}}_X^{\log}$ determined by sending each t^{-a} with $a \in P^{\text{gp}}$ to $1 \in \tilde{\tau}^{-1} \mathcal{O}_X \subseteq \tilde{\mathcal{O}}_X^{\log}$. This induces a ring homomorphism $A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0] \rightarrow \tilde{\mathcal{O}}_X^{\log} \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0] = \tilde{\mathcal{O}}_{\Lambda_0}^{\log}$. The tensor product

$$\tilde{\Psi}(E) := \tilde{E} \otimes_{A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]} \tilde{\mathcal{O}}_{\Lambda_0}^{\log}$$

has a structure of $\mathbb{Z}(P)$ -equivariant sheaf of $\tilde{\mathcal{O}}_{\Lambda_0}^{\log}$ -modules. This last operation has the effect of imposing that the action of P^{gp} is trivial (i.e. it identifies $e_\lambda \in E_\lambda$ with the image $e_\lambda \otimes t^{-p} \in E_\lambda \otimes_{\mathcal{O}_X} L_p \cong E_{\lambda+p}$ for $p \in P^{\text{gp}}$), and of adding the coefficients of \mathcal{O}_X^{\log} .

Remark 5.2. Imposing that the action of P^{gp} is trivial might look strange, but should be compared with the following situation for root stacks: as recalled in (3.4), given a parabolic sheaf E with weights in the Kummer extension $P \rightarrow Q$, to obtain a quasi-coherent sheaf on the root stack

$$\sqrt[q]{X} = [(X \times_{\text{Spec } \mathbb{Z}[P]} \text{Spec } \mathbb{Z}[Q]) / \mu_{Q/P}] \cong [\text{Spec}_X(\mathcal{O}_X[P^{\text{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) / \hat{Q}]$$

one forms the sheaf $\bigoplus_{q \in Q^{\text{gp}}} E_q$ on X , and equips it with a structure of Q^{gp} -graded $\mathcal{O}_X[P^{\text{gp}}] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ -module.

However, the presentation that we are using for the Kato-Nakayama space is closer to the first expression of $\sqrt[Q]{X}$ as a quotient stack, and the way to obtain a sheaf for that presentation is to pullback along the zero section $X \rightarrow \underline{\text{Spec}}_X(\mathcal{O}_X[P^{\text{gp}}]) = X \times \hat{P}$. This corresponds to forcing the action of P^{gp} to be trivial.

By descent along $\pi: \tilde{X}_{\log} \rightarrow X_{\log}$, this gives us a sheaf of $\mathcal{O}_{\Lambda_0}^{\log}$ -modules on X_{\log} . Now observe that the action of $\mathcal{O}_{\Lambda_0}^{\log}$ factors through $\mathcal{O}_{\Lambda}^{\log}$: if λ is a local section in the kernel of $(\Lambda_0)_X \rightarrow \Lambda$, then the action of $t^\lambda \in \mathcal{O}_{\Lambda_0}$ is given by the identity on the pieces of the parabolic sheaf E , and hence also on the sheaf \tilde{E} . Denote the resulting sheaf of $\mathcal{O}_{\Lambda}^{\log}$ -modules by $\Psi(E)$. Again assuming that we have proven that Ψ sends finitely presented parabolic sheaves to finitely presented sheaves, it is clear that if E is a quasi-coherent parabolic sheaf, then $\Psi(E)$ is quasi-coherent.

This defines the quasi-inverse on objects. It is straightforward to define the action of the functor $\Psi: \text{Par}(X, \Lambda) \rightarrow \text{Qcoh}_{\Lambda}(X)$ on arrows.

Ψ is a quasi-inverse:

Let us check that, still in the case where there is a Kato chart $X \rightarrow \mathbb{C}(P)$, the functors Φ and Ψ are quasi-inverses. Consider a parabolic sheaf $E \in \text{Par}(X, \Lambda_0)$, and the parabolic sheaf $\Phi(\Psi(E))$. For every $\lambda \in \Lambda_0^{\text{wt}}$, the sheaf $\Phi(\Psi(E))_\lambda$ is the pushforward $\tau_*(\Psi(E) \otimes_{\mathcal{O}_{\Lambda_0}^{\log}} L_\lambda^{\Lambda_0}) \in \text{Qcoh}(X)$. We can also compute this as $\tilde{\tau}_*(\tilde{\Psi}(E) \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0})$.

Note that there is a natural \mathcal{O}_X -linear injective morphism

$$E_\lambda \rightarrow \tilde{\tau}_*(\tilde{\Psi}(E) \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0}),$$

that sends $f \in E_\lambda$ to the section

$$(f \otimes 1) \otimes t^{-\lambda} \in \left(\left(\bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} E_\lambda \right) \otimes_{A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]} \tilde{\mathcal{O}}_{\Lambda_0}^{\log} \right) \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0}.$$

We claim that this map is an isomorphism.

Let $s = \sum_i (a_i \otimes b_i) \otimes c_i$ be a section of $\tilde{\tau}_*(\tilde{\Psi}(E) \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0})$, seen as a $\mathbb{Z}(P)$ -invariant section of $\tilde{\Psi}(E) \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0}$ on \tilde{X}_{\log} . In particular a_i are “homogeneous” sections of \tilde{E} (i.e. in some E_{λ_i}), b_i are sections of $\tilde{\mathcal{O}}_{\Lambda_0}^{\log}$, and c_i are sections of $\tilde{L}_\lambda^{\Lambda_0}$. By bilinearity and by moving the coefficients to the other factors, we can assume that $c_i = t^{-\lambda}$ (the local generator of the line bundle) for every i . Moreover, it is clear that if s is $\mathbb{Z}(P)$ -invariant, then b_i has to be in $\tilde{\mathcal{O}}_{\Lambda_0} \subseteq \tilde{\mathcal{O}}_{\Lambda_0}^{\log}$ for every i (recall that locally $\tilde{\mathcal{O}}_X^{\log}$ is a polynomial ring with coefficients in $\tilde{\tau}^{-1}\mathcal{O}_X$, and $\mathbb{Z}(P)$ acts on each “indeterminate” by adding integer multiples of $2\pi i$). By moving coefficients to the first factor, we can assume that $b_i = 1$ for every i .

Hence we are reduced to a section of the form $\sum_i (a_i \otimes 1) \otimes t^{-\lambda}$. By the explicit form of the action of $\mathbb{Z}(P)$, it is clear that this is invariant if and only if $e^{2\pi i g_{\mathbb{R}}(\lambda_i - \lambda)} = 1$ for every i and every $g \in \mathbb{Z}(P)$ (where $a_i \in E_{\lambda_i}$). This implies that $\lambda_i - \lambda$ is zero in $\Lambda_0^{\text{gp}}/P^{\text{gp}}$ for all i , i.e. there exist $m_i \in P^{\text{gp}}$ such that $\lambda_i - \lambda = m_i$ for all i . Finally, we claim that each term $(a_i \otimes 1) \otimes t^{-\lambda}$ is equal to some $(d_i \otimes 1) \otimes t^{-\lambda}$ with $d_i \in E_\lambda$. Indeed, since $\lambda_i - \lambda = m_i \in P^{\text{gp}}$, by acting on a_i via P^{gp} we can obtain a section of E_λ . But by construction, the action of P^{gp} on $\tilde{\Psi}(E)$ is the identity.

This gives an isomorphism $\Phi(\Psi(E))_\lambda = \tilde{\tau}_*(\tilde{E} \otimes_{\tilde{\mathcal{O}}_{\Lambda_0}^{\log}} \tilde{L}_\lambda^{\Lambda_0}) \cong E_\lambda$ for every $\lambda \in \Lambda_0$. By how the action of the section s_λ of $L_\lambda^{\Lambda_0}$ is defined on $\Psi(E)$ it is also clear that the map $E_\lambda \cong \Phi(\Psi(E))_\lambda \rightarrow \Phi(\Psi(E))_{\lambda'} \cong E_{\lambda'}$ coincides with the given one $E_\lambda \rightarrow E_{\lambda'}$ for each $\lambda \leq \lambda'$ in Λ^{wt} . After easily checking that these isomorphisms are compatible with restrictions to open subsets and functorial in the parabolic sheaf E , we conclude that there is a functorial isomorphism of parabolic sheaves $\Phi(\Psi(E)) \cong E$.

Conversely, let us start from a quasi-coherent sheaf F on X_{\log} , and show that there is a natural isomorphism $F \cong \Psi(\Phi(F))$. For this we can pull everything back along $\pi: \tilde{X}_{\log} \rightarrow X_{\log}$, and check that $\pi^{-1}F \cong \tilde{\Psi}(\Phi(F))$. Note that in fact there is a functorial morphism $\tilde{\Psi}(\Phi(G)) \rightarrow \pi^{-1}G$ for $G \in \text{Qcoh}_\Lambda(X)$, obtained from the natural maps $\tilde{\tau}^{-1}\Phi(G)_\lambda \rightarrow \pi^{-1}G$. By further localizing on X we can assume that F is a filtered colimit of finitely presented sheaves, and thus it suffices to prove that claim for F finitely presented.

Assume that F has a presentation

$$\bigoplus_j L_{\lambda_j}^\Lambda \longrightarrow \bigoplus_i L_{\lambda_i}^\Lambda \longrightarrow F \longrightarrow 0$$

with finitely many summands, that we can pull back to \tilde{X}_{\log} , obtaining a presentation

$$\bigoplus_j \tilde{L}_{\lambda_j}^\Lambda \longrightarrow \bigoplus_i \tilde{L}_{\lambda_i}^\Lambda \longrightarrow \pi^{-1}F \longrightarrow 0.$$

Recall moreover that $\tilde{\Psi}(\Phi(F)) = \left(\bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} \Phi(F)_\lambda \right) \otimes_{A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]} \tilde{\mathcal{O}}_{\Lambda_0}^{\log}$, and $\Phi(F)_\lambda = \tau_*(F \otimes_{\mathcal{O}_{\Lambda_0}^{\log}} L_\lambda^{\Lambda_0})$.

From the exactness of the various functors we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \tilde{\Psi}(\Phi(\bigoplus_j L_{\lambda_j}^\Lambda)) & \longrightarrow & \tilde{\Psi}(\Phi(\bigoplus_i L_{\lambda_i}^\Lambda)) & \longrightarrow & \tilde{\Psi}(\Phi(F)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_j \tilde{L}_{\lambda_j}^\Lambda & \longrightarrow & \bigoplus_i \tilde{L}_{\lambda_i}^\Lambda & \longrightarrow & \pi^{-1}F & \longrightarrow & 0. \end{array}$$

Finally, it is easy to see that the two leftmost vertical maps are isomorphisms, and hence also the right map is, as we wanted to prove.

Exactness and finite presentation:

It is clear from exactness of τ_* (Proposition 4.19) that Φ respects exactness. Let us show that it restricts to the subcategories of finitely presented sheaves on both sides.

Assume that F is a finitely presented sheaf of $\mathcal{O}_\Lambda^{\log}$ -modules on X_{\log} , as in Definition 4.15. By localizing on X we can assume that we have a presentation

$$\bigoplus_j L_{\lambda_j}^\Lambda \longrightarrow \bigoplus_i L_{\lambda_i}^\Lambda \longrightarrow F \longrightarrow 0$$

with finitely many summands, and that we have charts $P \rightarrow \overline{M}(X)$ and $\Lambda_0 \rightarrow \Lambda(X)$. By Proposition 4.22 all the pieces $\Phi(F)_\lambda$ are finitely presented sheaves on X . Moreover, consider the sub-weight system $R \subseteq \Lambda_0^{\text{wt}}$ given by the orbits for the P^{gp} -action of the (finitely many) elements λ_i and λ_j appearing in the presentation above. We claim that $\Phi(F)$ is the induction of a parabolic sheaf with weights in R .

Specifically, we claim that $\Phi(F)$ is isomorphic to $\text{Ind}_R^{\Lambda_0^{\text{wt}}}(G)$, where G is the sheaf $\text{Res}_R^{\Lambda_0^{\text{wt}}}(\Phi(F))$. To verify this, it is enough to prove that for every $\lambda \in \Lambda_0$ the map

$$\varinjlim_{R \ni r \leq \lambda} \Phi(F)_r \rightarrow \Phi(F)_\lambda$$

is an isomorphism.

By applying the two functors to the presentation of F above (which stays exact), we see that it is enough to check the statement for the sheaves $\bigoplus_i L_{\lambda_i}^\Lambda$ and $\bigoplus_j L_{\lambda_j}^\Lambda$, for which it is an easy computation.

Conversely, assume that E is a finitely presented parabolic sheaf on X , and let us show that $\Psi(E)$ is finitely presented on X_{\log} . As above, we can localize on X where there are charts $P \rightarrow \overline{M}(X)$ and $\Lambda_0 \rightarrow \Lambda(X)$, and where E comes via induction from a finite sub-system $R \subseteq \Lambda_0^{\text{wt}}$. We can also assume that each of the (finitely many) sheaves E_r with $r \in \Lambda_0 \setminus \langle P^+ \rangle$ (here P^+ denotes $P \setminus \{0\}$) admits a presentation as

$$\mathcal{O}_X^{\oplus J_r} \xrightarrow{f_r} \mathcal{O}_X^{\oplus I_r} \longrightarrow E_r \longrightarrow 0$$

where I_r and J_r are finite sets. It is easy to check that the sheaf $\tilde{\Psi}(E)$ on \tilde{X}_{\log} has a presentation of the form

$$\bigoplus_r (\tilde{L}_{-r}^\Lambda)^{\oplus J_r} \xrightarrow{\oplus (f_r)_{-r}} \bigoplus_r (\tilde{L}_{-r}^\Lambda)^{\oplus I_r} \longrightarrow \tilde{\Psi}(E) \longrightarrow 0$$

which shows that $\Psi(E)$ is finitely presented. The map $\bigoplus_r (\tilde{L}_{-r}^\Lambda)^{\oplus I_r} \rightarrow \tilde{\Psi}(E)$ is defined by mapping the factor $(\tilde{L}_{-r}^\Lambda)^{\oplus I_r}$ into $\tilde{\tau}^{-1}E_r$ via the map coming from the presentation above (recall that $\tilde{\Psi}(E) = (\bigoplus_{\lambda \in \Lambda_0^{\text{gp}}} \tilde{\tau}^{-1}E_\lambda) \otimes_{A \otimes_{\mathbb{C}[P]} \mathbb{C}[\Lambda_0]} \tilde{\mathcal{O}}_{\Lambda_0}^{\log}$).

This concludes the proof. \square

Remark 5.3. One can check, using the construction of Φ , that for two quasi-coherent sheaves of monoids $\overline{M} \subseteq \Lambda \subseteq \Lambda' \subseteq \overline{M}_{\mathbb{R}}$, with induced weight systems $W = \Lambda^{\text{wt}}$ and $W' = (\Lambda')^{\text{wt}}$, the restriction $\text{Res}_W^{W'}$ and induction $\text{Ind}_W^{W'}$ on parabolic sheaves correspond respectively to equipping a sheaf of $\mathcal{O}_{\Lambda'}^{\log}$ -modules with the structure of $\mathcal{O}_{\Lambda}^{\log}$ -module coming from the natural map $\mathcal{O}_{\Lambda}^{\log} \rightarrow \mathcal{O}_{\Lambda'}^{\log}$, and to taking the tensor product $- \otimes_{\mathcal{O}_{\Lambda}^{\log}} \mathcal{O}_{\Lambda'}^{\log}$.

5.1. Comparison with root stacks. To conclude, we compare the equivalence of Theorem 5.1 to the ones between parabolic sheaves and sheaves on root stacks, of [5] and [22].

Let X be a fine saturated log scheme locally of finite type over \mathbb{C} . Then for every n there is a canonical morphism of topological stacks $\phi_n: X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$, coming for example from the fact that the projection $\sqrt[n]{X} \rightarrow X$ induces an isomorphism $(\sqrt[n]{X})_{\log} \rightarrow X_{\log}$ (see the proof of [23, Proposition 4.6]). These are compatible for different indices, and induce a morphism $\phi_\infty: X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$ (see [6, Proposition 4.1] or [23, Section 3.4]).

By [5, Theorem 6.1] and [22, Theorem 7.3] we have compatible equivalences of abelian categories $\Phi_n: \text{Qcoh}(\sqrt[n]{X}) \rightarrow \text{Par}(X, \frac{1}{n}M)$ and $\Phi_\infty: \text{Qcoh}(\sqrt[\infty]{X}) \rightarrow \text{Par}(X, \mathbb{Q})$. We will prove that these are compatible with the equivalence Φ of Theorem 5.1, in the following sense. We have several structures on the root stacks that we can pull back via the morphisms ϕ_n .

The stack $\sqrt[n]{X}_{\text{top}}$ has a structure sheaf \mathcal{O}_n , and a tautological DF structure $L_n: \frac{1}{n}\overline{M} \rightarrow \text{Div}_{\sqrt[n]{X}}$ (if $n = \infty$, then $\frac{1}{n}\overline{M}$ denotes $M_{\mathbb{Q}}$). These are all compatible with respect to pullbacks along the projections $\sqrt[m]{X} \rightarrow \sqrt[n]{X}$. In the case of the infinite root stack, it is better to think about \mathcal{O}_∞ as $\varinjlim_n \mathcal{O}_n$, so that $\phi_\infty^* \mathcal{O}_\infty$ on the space X_{\log} is $\varinjlim_n \phi_n^* \mathcal{O}_n$.

Here when we write $\phi_n^* F$ for a sheaf F of \mathcal{O}_n -modules on $\sqrt[n]{X}_{\text{top}}$, we mean the sheaf $\phi_n^{-1} F \otimes_{\tau^{-1} \mathcal{O}_X} \mathcal{O}_X^{\log}$ on X_{\log} .

Proposition 5.4. *There is a sequence of compatible isomorphisms of sheaves of rings $\phi_n^* \mathcal{O}_n \cong \mathcal{O}_{\frac{1}{n}\overline{M}}^{\log}$, where we are using the notation of (4.3) for the sheaf on the right hand side. Moreover, the pullback DF structure $\phi_n^* L_n: \frac{1}{n}\overline{M} \rightarrow \text{Div}_{\mathcal{O}_{\frac{1}{n}\overline{M}}^{\log}}$ is canonically isomorphic to the DF structure described in (4.3).*

Proof. We can reduce to checking the claim on log schemes of the form $\text{Spec } \mathbb{C}[P]$ for a toric monoid P . Moreover, it is enough to prove that $\phi_n^{-1} \mathcal{O}_n \cong \mathcal{O}_{\frac{1}{n}\overline{M}}$.

As briefly explained in (2.3), we have a diagram

$$\begin{array}{ccc} \tilde{X}_{\log} \times \mathbb{Z}(P) & \longrightarrow & \mathbb{C}(\frac{1}{n}P) \times \mu_n(P) \\ \downarrow \downarrow & & \downarrow \downarrow \\ \tilde{X}_{\log} & \xrightarrow{\tilde{\phi}_n} & \mathbb{C}(\frac{1}{n}P) \\ \downarrow & & \downarrow \\ X_{\log} & \xrightarrow{\phi_n} & \sqrt[n]{\mathbb{C}(P)} \end{array}$$

where $\tilde{\phi}_n: \tilde{X}_{\log} = \text{Hom}(P, \mathbb{H}) \rightarrow \mathbb{C}(\frac{1}{n}P) = \text{Hom}(\frac{1}{n}P, \mathbb{C})$ is given by composing $\frac{1}{n}P \cong P \rightarrow \mathbb{H}$ with $\mathbb{H} \rightarrow \mathbb{H} \rightarrow \mathbb{C}$, where the first map is $(x, y) \mapsto (\sqrt[n]{x}, y/n)$ and the second map is $(x, y) \mapsto x \cdot e^{iy}$. The homomorphism $\mathbb{Z}(P) = \text{Hom}(P, \mathbb{Z}) \rightarrow \mu_n(P) = \text{Hom}(P, \mathbb{Z}/n\mathbb{Z})$ is also defined by composing with $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, and $\tilde{\phi}_n$ is equivariant with respect to this homomorphism.

Hence we can show that $\tilde{\phi}_n^{-1} \mathcal{O}_n \cong \tilde{\mathcal{O}}_{\frac{1}{n}\overline{M}}$ as $\mathbb{Z}(P)$ -equivariant sheaves. This is clear from the fact that $\tilde{\mathcal{O}}_{\frac{1}{n}\overline{M}}$ is the quotient of $\tilde{\tau}^{-1} \mathcal{O}_X \times_{\mathbb{C}[P]} \mathbb{C}[\frac{1}{n}P]$ by the ideal I generated by local sections of the form $t^a - 1$, where $a \in \frac{1}{n}P$ maps to zero in $\overline{M}_{\mathbb{R}}$: we have a natural map $\tilde{\tau}^{-1} \mathcal{O}_X \times_{\mathbb{C}[P]} \mathbb{C}[\frac{1}{n}P] \rightarrow \tilde{\phi}_n^{-1} \mathcal{O}_n$ induced by $\tilde{\tau}^{-1} \mathcal{O}_X = \tilde{\phi}_n^{-1} \pi_n^{-1} \mathcal{O}_X \rightarrow \tilde{\phi}_n^{-1} \mathcal{O}_n$ and $\mathbb{C}[\frac{1}{n}P] \rightarrow \mathcal{O}_n$, which factors through the quotient by the sheaf of ideals I . One can verify that the resulting map is an isomorphism, for example by looking at the stalks.

The assertion about the DF structure is proved similarly. \square

Remark 5.5. The preceding discussion explains [23, Remark 4.7]: if we consider for a log algebraic stack X its Kato-Nakayama space X_{\log} as a ringed topological stack, equipped with the sheaf of rings \mathcal{O}_X^{\log} , then the isomorphism $(\sqrt[n]{X})_{\log} \cong X_{\log}$ is not an isomorphism of ringed topological stacks, since the structure sheaf of $(\sqrt[n]{X})_{\log}$ is identified with the sheaf $\mathcal{O}_{\frac{1}{n}\overline{M}}^{\log}$ on X_{\log} .

Let us now check that the equivalence Φ of Theorem 5.1 is compatible with the analogous equivalences on the root stacks.

Proposition 5.6. *The following diagram of functors is 2-commutative.*

$$\begin{array}{ccc} \text{Qcoh}(\sqrt[n]{X}) & \xrightarrow{\phi_n^*} & \text{Qcoh}_{\frac{1}{n}\overline{M}}(X_{\log}) \\ & \searrow \Phi_n \quad \swarrow \Phi & \\ & \text{Par}(X, \frac{1}{n}\overline{M}) & \end{array}$$

Moreover all the functors restrict to the subcategories of finitely presented sheaves, and the diagrams for different n are compatible with respect to pushforward and pullback along projections between root stacks, and induction and restriction between the categories of parabolic sheaves.

In particular, for every n the pullback functor ϕ_n^* is an equivalence.

Proof. We can assume that X has a global Kato chart $X \rightarrow \text{Spec } \mathbb{C}[P]$. Fix a quasi-coherent sheaf $F \in \text{Qcoh}(\sqrt[n]{X})$, and an element $\frac{1}{n}a \in \frac{1}{n}P^{\text{gp}}$. We have to check that $\Phi(\phi_n^*F)$ and Φ_n are the same parabolic sheaf on X with weights in $\frac{1}{n}P$.

We have

$$\Phi_n(F)_{\frac{1}{n}a} = (\pi_n)_*(F \otimes_{\mathcal{O}_n} L_{\frac{1}{n}a}),$$

where $\pi_n: \sqrt[n]{X} \rightarrow X$ is the projection and $L: \frac{1}{n}P^{\text{gp}} \rightarrow \text{Pic}(\sqrt[n]{X})$ is the symmetric monoidal functor corresponding to the universal DF structure on the root stack. On the other hand

$$\Phi(\phi_n^*F)_{\frac{1}{n}a} = \tau_*(\phi_n^*F \otimes_{\mathcal{O}_{\frac{1}{n}\overline{M}}} L_{\frac{1}{n}a}^{\frac{1}{n}\overline{M}}),$$

where $L_{\frac{1}{n}\overline{M}}: \frac{1}{n}P^{\text{gp}} \rightarrow \text{Pic}(X_{\log}, \mathcal{O}_{\frac{1}{n}\overline{M}}^{\log})$ is the analogous DF structure on the ringed space $(X_{\log}, \mathcal{O}_{\frac{1}{n}\overline{M}}^{\log})$.

Now note that $L_{\frac{1}{n}a}^{\frac{1}{n}\overline{M}} \cong \phi_n^*L_{\frac{1}{n}a}$, so that we have

$$\Phi(\phi_n^*F)_{\frac{1}{n}a} = \tau_*(\phi_n^*(F \otimes_{\mathcal{O}_n} L_{\frac{1}{n}a})).$$

Finally, since $(X_{\log}, \mathcal{O}_{\frac{1}{n}\overline{M}}^{\log}) = ((\sqrt[n]{X})_{\log}, \mathcal{O}_{\sqrt[n]{X}}^{\log}) \rightarrow \sqrt[n]{X}$ can be seen as the projection from the Kato-Nakayama space of $\sqrt[n]{X}$ and $\tau_* = (\pi_n)_* \circ (\phi_n)_*$, the analogue of Proposition 4.23 (whose proof we leave to the reader) implies that $(\phi_n)_* \circ \phi_n^* \cong \text{id}$, and hence $\Phi_n(F)_{\frac{1}{n}a} \cong \Phi(\phi_n^*F)_{\frac{1}{n}a}$.

The remaining assertions are routinely checked. \square

Remark 5.7. Let us also briefly comment on the relationship between our setup and the similar one of [8]. In Section 3 of that paper, the authors consider the Kummer-étale topos X_{ket} , which is equipped with a natural morphism $X_{\text{ket}} \rightarrow X_{\text{an}}$, and construct a morphism of topoi $\tau_{\text{ket}}: X_{\log} \rightarrow X_{\text{ket}}$ factoring $\tau: X_{\log} \rightarrow X_{\text{an}}$. They consider a sheaf of rings $\mathcal{O}_X^{\text{klog}}$ on X_{\log} , defined as $\tau_{\text{ket}}^{-1}\mathcal{O}_X^{\text{ket}} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_X^{\log}$, where $\mathcal{O}_X^{\text{ket}}$ is the structure sheaf of the Kummer-étale topos. In (3.2), the authors give a description of $\mathcal{O}_X^{\text{klog}}$ that coincides with the one of Remark 4.13, seeing it as being generated over \mathcal{O}_X by formal logarithms (which correspond to \mathcal{O}_X^{\log}) and formal n -th roots for every n (which correspond to $\mathcal{O}_X^{\text{ket}}$), of sections of \overline{M} .

The ringed topos $(X_{\text{ket}}, \mathcal{O}_X^{\text{klog}})$ is equivalent to the infinite root stack, in the following sense. The natural functor from the Kummer-étale site that takes a Kummer-étale morphism $U \rightarrow X$ to the map between infinite root stacks $\sqrt[n]{U} \rightarrow \sqrt[n]{X}$ induces a morphism of sites from an opportunely defined small étale site of $\sqrt[n]{X}$ to the Kummer-étale site of X , which gives a morphism of topoi $\rho: \sqrt[n]{X} \rightarrow X_{\text{ket}}$ (where we identify the stack $\sqrt[n]{X}$ with its small étale topos), which is proved to be an equivalence in [22, Theorem 6.21].

In fact, as mentioned above, there is a natural isomorphism $\mathcal{O}_X^{\text{klog}} \cong \mathcal{O}_{\overline{M}_{\mathbb{Q}}}^{\log}$, and, by Proposition 5.4, the last sheaf can also be seen as the pullback $\phi_{\infty}^*\mathcal{O}_{\infty}$ along $\phi_{\infty}: X_{\log} \rightarrow \sqrt[n]{X}$.

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